K-stability and Kähler-Einstein metrics

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1 Introduction

In this paper, we solve a folklore conjecture [1] on Fano manifolds without non-trivial holomorphic vector fields. The main technical ingredient is a conic version of Cheeger-Colding-Tian's theory on compactness of Kähler-Einstein manifolds. This enables us to prove an extension of the partial C^0-estimate for Kähler-Einstein metrics established in [DS12] and [Ti12].

A Fano manifold is a projective manifold with positive first Chern class $c_1(M)$. Its holomorphic fields form a Lie algebra $\eta(M)$. The folklore conjecture states: *If $\eta(M) = \{0\}$, then M admits a Kähler-Einstein metric if and only if M is K-stable with respect to the anti-canonical bundle K_M^{-1}.* Its necessary part was established in [Ti97]. The following gives the sufficient part of this conjecture.

Theorem 1.1. *Let M be a Fano manifold canonically polarized by the anti-canonical bundle K_M^{-1}. If M is K-stable, then it admits a Kähler-Einstein metric.*

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[1] It is often referred as the Yau-Tian-Donaldson conjecture

An older approach for proving this theorem is to solve the following complex Monge-Ampere equations by the continuity method:

$$(\omega + \sqrt{-1}\,\partial\bar{\partial}\varphi)^n \,=\, e^{h-t\varphi}\omega^n, \quad \omega + \sqrt{-1}\,\partial\bar{\partial}\varphi \,>\, 0, \tag{1.1}$$

where ω is a given Kähler metric with its Kähler class $[\omega] = 2\pi c_1(M)$ and h is uniquely determined by

$$\mathrm{Ric}(\omega) - \omega \,=\, \sqrt{-1}\,\partial\bar{\partial}h, \quad \int_M (e^h - 1)\,\omega^n \,=\, 0.$$

Let I be the set of t for which (1.1) is solvable. Then we have known: (1) By the well-known Calabi-Yau theorem, I is non-empty; (2) In 1983, Aubin proved that I is open [Au83]; (3) If we can have an a priori C^0-estimate for the solutions of (1.1), then I is closed and consequently, there is a Kähler-Einstein metric on M.

However, the C^0-estimate does not hold in general since there are many Fano manifolds which do not admit any Kähler-Einstein metrics. The existence of Kähler-Einstein metrics required certain geometric stability on the underlying Fano manifolds. In early 90's, I proposed a program towards establishing the existence of Kähler-Einstein metrics. The key technical ingredient of this program is a conjectured partial C^0-estimate. If we can affirm this conjecture for the solutions of (1.1), then we can use the K-stability to derive the a prior C^0-estimate and the Kähler-Einstein metric. The K-stability was first introduced in [Ti97] as a test for the properness of the K-energy restricted to a finite dimensional family of Kähler metrics induced by a fixed embedding by pluri-anti-canonical sections.[2] However, such a conjecture on partial C^0-estimates is still open except for Kähler-Einstein metrics.

In [Do10], Donaldson suggested a new continuity method by using conic Kähler-Einstein metrics. Those are metrics with conic angle along a divisor. For simplicity, here we consider only the case of smooth divisors.

Let M be a compact Kähler manifold and $D \subset M$ be a smooth divisor. A conic Kähler metric on M with angle $2\pi\beta$ $(0 < \beta \leq 1)$ along D is a Kähler metric on $M \backslash D$ that is asymptotically equivalent along D to the model conic metric

$$\omega_{0,\beta} \,=\, \sqrt{-1}\left(\frac{dz_1 \wedge d\bar{z}_1}{|z_1|^{2-2\beta}} + \sum_{j=2}^{n} dz_j \wedge d\bar{z}_j\right),$$

where $z_1, z_2, \cdots, z_n$ are holomorphic coordinates such that $D = \{z_1 = 0\}$ locally. Each conic Kähler metric can be given by its Kähler form ω which represents a cohomology class in $H^{1,1}(M,\mathbb{C}) \cap H^2(M,\mathbb{R})$, referred as the Kähler class $[\omega]$. A conic Kähler-Einstein metric is a conic Kähler metric which is also Einstein outside conic points.

In this paper, we only need to consider the following conic Kähler-Einstein metrics: Let M be a Fano manifold and D be a smooth divisor which represents

[2]The K-stability was reformulated in more algebraic ways (see [Do02], [Pa08] et al.).

the Poincare dual of $\lambda c_1(M)$. We call ω a conic Kähler-Einstein with conic angle $2\pi\beta$ along D if it has $2\pi c_1(M)$ as its Kähler class and satisfies

$$\mathrm{Ric}(\omega) \,=\, \mu\omega \,+\, 2\pi(1-\beta)[D]. \tag{1.2}$$

Here the equation on M is in the sense of currents, while it is classical outside D. We will require $\mu > 0$ which is equivalent to $(1-\beta)\lambda < 1$. As in the smooth case, each conic Kähler metric ω with $[\omega] = 2\pi c_1(M)$ is the curvature of a Hermitian metric $||\cdot||$ on the anti-canonical bundle K_M^{-1}. The difference is that the Hermitian metric is not smooth, but it is Hölder continuous.

Donaldson's continuity method was originally proposed as follows: Assume that $\lambda = 1$, i.e., D be a smooth anti-canonical divisor. It follows from [TY90] that there is a complete Calabi-Yau metric on $M\backslash D$. It was conjectured that this complete metric is the limit of Kähler-Einstein metrics with conic angle $2\pi\beta \mapsto 0$. If this is true, then the set E of $\beta \in (0,1]$ such that there is a conic Kähler metric satisfying (1.2) is non-empty. It is proved in [Do10] that E is open. Then we are led to proving that E is closed.

A problem with this original approach of Donaldson arose because we do not know if a Fano manifold M always has a smooth anti-canonical divisor D. Possibly, there are Fano manifolds which do not admit smooth anti-canonical divisors. At least, it seems to be a highly non-trivial problem whether or not any Fano manifold admits a smooth anti-canonical divisor. Fortunately, Li and Sun bypassed this problem. Inspired by [JMR11], they modified Donaldson's original approach by allowing $\lambda > 1$. They observed that the main existence theorem in [JMR11], coupled with an estimate on log-α invariants in [Be11], implies the existence of conic Kähler-Einstein metrics with conic angle $2\pi\beta$ so long as $\mu = 1-(1-\beta)\lambda$ is sufficiently small. Now we define E to be set of $\beta \in (1-\lambda^{-1}, 1]$ such that there is a conic Kähler metric satisfying (1.2). Then E is non-empty. It follows from [Do10] that E is open. The difficult part is to prove that E is closed.

The construction of Kähler-Einstein metrics with conic angle $2\pi\beta$ can be reduced to solving complex Monge-Ampere equations:

$$(\omega_\beta \,+\, \sqrt{-1}\partial\bar\partial\varphi)^n \,=\, e^{h_\beta - \mu\varphi}\omega_\beta^n, \tag{1.3}$$

where ω_β is a suitable family of conic Kähler metrics with $[\omega_\beta] = 2\pi c_1(M)$ and cone angle $2\pi\beta$ along D and h_β is determined by

$$\mathrm{Ric}(\omega_\beta) \,=\, \mu\,\omega_\beta \,+\, 2\pi(1-\beta)\,[D] + \sqrt{-1}\,\partial\bar\partial h_\beta \quad\text{and}\quad \int_M (e^{h_\beta}-1)\,\omega_\beta^n \,=\, 0.$$

As shown in [JMR11], it is crucial for solving (1.3) to establish an a priori C^0-estimate for its solutions. Such a C^0-estimate does not hold in general. Therefore, following my program on the existence of Kähler-Einstein metrics through the Aubin's continuity method, we can first establish a partial C^0-estimate and then use the K-stability to conclude the C^0-estimate, consequently, the existence of Kähler-Einstein metrics on Fano manifolds which are K-stable.

For any integer $\lambda > 0$ and $\beta > 0$, let $\mathcal{E}(\lambda, \beta)$ be the set of all triples (M, D, ω), where M is a Fano manifold, D is a smooth divisor whose Poincare dual is $\lambda\, c_1(M)$ and ω is a conic Kähler-Einstein metric on M with cone angle $2\pi\beta$ along D. For any $\omega \in \mathcal{E}(\lambda, \beta)$, choose a C^1-Hermitian metric h with ω as its curvature form and any orthonormal basis $\{S_i\}_{0 \le i \le N}$ of each $H^0(M, K_M^{-\ell})$ with respect to the induced inner product by h and ω. Then as did in the smooth case, we can introduce a function

$$\rho_{\omega,\ell}(x) \,=\, \sum_{i=0}^{N} ||S_i||_h^2(x). \tag{1.4}$$

One of main results in this paper is the following.

Theorem 1.2. *For any fixed λ and $\beta_0 > 1 - \lambda^{-1}$, there are uniform constants $c_k = c(k, n, \lambda, \beta_0) > 0$ for $k \ge 1$ and $\ell_i \to \infty$ such that for any $\beta \ge \beta_0$ and $\omega \in \mathcal{E}(\lambda, \beta)$, we have for $\ell = \ell_i$,*

$$\rho_{\omega,\ell} \,\ge\, c_\ell \,> 0. \tag{1.5}$$

In [Ti12], we conjectured that this theorem holds for more general conic Kähler metrics.[3]

The most crucial tool in proving Theorem 1.2 is an extension of a compactness theorem of Cheeger-Colding-Tian for Kähler-Einstein metrics. One needs extra technical inputs to establish such an extension.

As a consequence of Theorem 1.2, we have

Theorem 1.3. *Let M be a Fano manifold with a smooth pluri-anti-canonical divisor D of $K_M^{-\lambda}$. Assume that ω_i be a sequence of conic Kähler-Einstein metrics with cone angle $2\pi\beta_i$ along D satisfying:*

$$\mathrm{Ric}(\omega_i) \,=\, \mu_i\omega_i \,+\, 2\pi(1-\beta_i)[D], \qquad \mu_i \,=\, 1-(1-\beta_i)\lambda.$$

where $\mu_i = 1 - (1-\beta_i)\lambda > 0$. We further assume that $\lim \mu_i = \mu_\infty > 0$ and (M, ω_i) converge to a length space (M_∞, d_∞) in the Gromov-Hausdorff topology. Then M_∞ is a smooth Kähler manifold outside a closed subset $\bar{S}$ of codimension at least 4 and d_∞ is induced by a smooth Kähler-Einstein metric outside a divisor $D_\infty \subset M_\infty$. Furthermore, (M, ω_i) converge to $(M_\infty, \omega_\infty)$ outside D_∞ in the C^∞-topology and D converges to D_∞ in the Gromov-Hausdorff topology.

This theorem is needed to finish the proof of Theorem 1.1.

The organization of this paper is as follows: In the next section, we prove an approximation theorem which states any conic Kähler-Einstein metrics can be approximated by smooth Kähler metrics with the same lower bound on Ricci curvature. This theorem was not known before and is of interest by itself. In section 3, we give an extension of my works with Cheeger-Colding in [CCT95] to

[3] Our method in this paper can be also applied to getting the partial C^0-estimate in this more general case.

conic Kähler-Einstein manifolds.[4] In section 4, we prove the smooth convergence for conic Kähler-Einstein metrics. In the smooth case, it is based on a result of M. Anderson. However, the arguments do not apply for the conic case. We have to introduce a new method. In Section 5, we prove Theorem 1.2, i.e., the partial C^0-estimate for conic Kähler-Einstein metrics. In last section, we prove Theorem 1.1.

The existence of Kähler-Einstein metrics on K-stable Fano manifold was first mentioned in my talk during the conference "Conformal and Kähler Geometry" held at IHP in Paris from September 17 to September 21 of 2012. On October 25 of 2012, in my talk at the Blainefest held at Stony Brook University, I outlined my proof of Theorem 1.1. I learned that X.X. Chen, S. Donaldson and S. Sun posted a short note on October 30 of 2012 in which they also announced a proof of Theorem 1.1.

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2 Smoothing conic Kähler-Einstein metrics

In this section, we address the question: *Can one approximate a conic Kähler-Einstein metrics by smooth Kähler metrics with Ricci curvature bounded from below*? For the sake of this paper, we confine ourselves to the case of positive scalar curvature. Our approach can be adapted to other cases where the scalar curvature is non-positive. In fact, the proof is even simpler.

Let ω be a conic Kähler-Einstein metric on M with cone angle $2\pi\beta$ along D, where D is a smooth divisor whose Poincare dual is equal to $\lambda\, c_1(M)$, in particular, ω satisfies (1.2) for $\mu = 1 - (1-\beta)\lambda > 0$. For any smooth Kähler metric ω_0 with $[\omega_0] = 2\pi c_1(M)$, we can write $\omega = \omega_0 + \sqrt{-1}\,\partial\bar\partial\varphi$ for some smooth function φ on $M\backslash D$. Note that φ is Hölder continuous on M. Define h_0 by

$$\mathrm{Ric}(\omega_0) \,-\, \omega_0 \,=\, \sqrt{-1}\,\partial\bar\partial h_0, \qquad \int_M (e^{h_0} - 1)\,\omega_0^n \,=\, 0.$$

[4] My work with Cheeger and Colding [CCT95] is definitely needed in establishing the partial C^0-estimate which is crucial in proving Theorem 1.1.

Note that the first equation above is equivalent to

$$\mathrm{Ric}(\omega_0) \,=\, \mu\,\omega_0 \,+\, 2\pi(1-\beta)[D] \,+\, \sqrt{-1}\,\partial\bar\partial(h_0 - (1-\beta)\,\log ||S||_0^2),$$

where S is a holomorphic section of $K_M^{-\lambda}$ defining D and $||\cdot||_0$ is a Hermitian norm on $K_M^{-\lambda}$ with $\lambda\,\omega_0$ as its curvature. For convenience, we assume that

$$\sup_M ||S||_0 = 1.$$

If ω_β and h_β are those in (1.3), then modulo a constant,

$$h_\beta \,=\, h_0 \,-\, (1-\beta)\log||S||_0^2 \,-\, \log\left(\frac{\omega_\beta^n}{\omega_0^n}\right) \,-\, \mu\,\psi_\beta,$$

where $\omega_\beta \,=\, \omega_0 \,+\, \sqrt{-1}\,\partial\bar\partial\psi_\beta$.

It follows from (1.2)

$$(\omega_0 + \sqrt{-1}\,\partial\bar\partial\varphi)^n \,=\, e^{h_0-(1-\beta)\log||S||_0^2 + a_\beta - \mu\varphi}\,\omega_0^n, \tag{2.1}$$

where a_β is chosen according to

$$\int_M \left(e^{h_0-(1-\beta)\log||S||_0^2+a_\beta} \,-\, 1\right)\omega_0^n \,=\, 0.$$

Clearly, a_β is uniformly bounded so long as $\beta \,\geq\, \beta_0 \,>\, 0$.

The Lagrangian $\mathbf{F}_{\omega_0,\mu}(\varphi)$ of (2.1) is given by

$$\mathbf{J}_{\omega_0}(\varphi) \,-\, \frac{1}{V}\int_M \varphi\,\omega_0^n \,-\, \frac{1}{\mu}\,\log\left(\frac{1}{V}\int_M e^{h_0-(1-\beta)\log||S||_0^2+a_\beta-\mu\varphi}\,\omega_0^n\right), \tag{2.2}$$

where $V = \int_M \omega_0^n$ and

$$\mathbf{J}_{\omega_0}(\varphi) \,=\, \frac{1}{V}\sum_{i=0}^{n-1}\frac{i+1}{n+1}\int_M \sqrt{-1}\,\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega_0^i\wedge\omega_\varphi^{n-i-1}, \tag{2.3}$$

where $\omega_\varphi = \omega_0 + \sqrt{-1}\,\partial\bar\partial\varphi$. Note that $\mathbf{F}_{\omega_0,\mu}$ is well-defined for any continuous function φ.

Let us recall the following result

Theorem 2.1. *If $\omega = \omega_\varphi$ is a conic Kähler-Einstein with conic angle $2\pi\beta$ along D, then φ attains the minimum of the functional $\mathbf{F}_{\omega_0,\mu}$ on the space $\mathcal{K}_\beta(M,\omega_0)$ which consists of all smooth functions ψ on $M\backslash D$ such that ω_ψ is a conic Kähler metric with angle $2\pi\beta$ along D. In particular, $\mathbf{F}_{\omega_0,\mu}$ is bounded from below.*

One can find its proof in [Bo11]. An alternative proof may be given by extending the arguments in [DT91] to conic Kähler metrics.

Corollary 2.2. *If $\mu < 1$, then there are $\epsilon > 0$ and $C_\epsilon > 0$, which may depend on ω and μ, such that for any $\psi \in \mathcal{K}_\beta(M, \omega_0)$, we have for any $t \in (0, \mu]$*[5]

$$\mathbf{F}_{\omega_0,t}(\psi) \,\geq\, \epsilon\, \mathbf{J}_{\omega_0}(\psi) \,-\, C_\epsilon. \tag{2.4}$$

Proof. It follows from the arguments of using the log-α-invariant in [LS12] that $\mathbf{F}_{\omega_0,t}$ is proper for $t > 0$ sufficiently small. Let $\omega = \omega_\varphi$ be the conic Kähler-Einstein metric with angle β along D. Then φ satisfies (2.1). Since M does not admit non-zero holomorphic fields,[6]it follows from [Do10] that (2.1) has a solution $\bar\varphi$ when μ is replaced by $\bar\mu \,=\, \mu + \delta$ for $\delta >$ sufficiently small. Hence, by Theorem 2.1, $\mathbf{F}_{\omega_0,\bar\mu}$ is bounded from below. Then this corollary follows from Proposition 1.1 in [LS12][7]

□

Now we consider the following equation:

$$(\omega_0 + \sqrt{-1}\,\partial\bar\partial\varphi)^n \,=\, e^{h_\delta - \mu\varphi}\,\omega_0^n, \tag{2.5}$$

where

$$h_\delta \,=\, h_0 - (1-\beta)\log(\delta + ||S||_0^2) + c_\delta$$

for some constant c_δ determined by

$$\int_M \left(e^{h_0-(1-\beta)\log(\delta+||S||_0^2)+c_\delta} - 1\right)\,\omega_0^n \,=\, 0.$$

Clearly, c_δ is uniformly bounded. If φ_δ is a solution, then we get a smooth Kähler metric

$$\omega_\delta \,=\, \omega_0 \,+\, \sqrt{-1}\,\partial\bar\partial\varphi_\delta.$$

Its Ricci curvature is given by

$$\mathrm{Ric}(\omega_\delta) \,=\, \mu\,\omega_\delta \,+\, \frac{\delta(1-\beta)\lambda}{\delta + ||S||_0^2}\,\omega_0 \,+\, \delta(1-\beta)\,\frac{DS\wedge\overline{DS}}{(\delta+||S||_0^2)^2},$$

where DS denotes the covariant derivative of S with respect to the Hermitian metric $||\cdot||_0$. In particular, the Ricci curvature of ω_δ is greater than μ whenever $\beta < 1$ and $\delta > 0$.[8]

We will solve (2.5) for such ω_δ's and show that they converge to the conic Kähler-Einstein metric ω in a suitable sense.

To solve (2.5), we use the standard continuity method:

$$(\omega_0 + \sqrt{-1}\,\partial\bar\partial\varphi)^n \,=\, e^{h_\delta - t\varphi}\,\omega_0^n. \tag{2.6}$$

[5]The corresponding β_t is defined by $(1-t) = (1-\beta_t)\lambda$.

[6]Even if M does have non-trivial holomorphic fields, there should be no holomorphic fields which are tangent to D. This is sufficient for rest of the proof.

[7]In [LS12], the reference metric ω_0 is a conic Kähler metric while ours is a smooth metric, however, the arguments apply with slight modification.

[8]This observation is crucial in our approximating the conic Kähler-Einstein metric ω and first appeared in the slides of my talk at SBU on October 25, 2012. The arguments in establishing the existence of ω_δ is known for long and identical to the one I used in [Ti97].

Define I_δ to be the set of $t \in [0, \mu]$ for which (2.6) is solvable. By the Calabi-Yau theorem, $0 \in I_\delta$.

We may assume $\mu < 1$, otherwise, we have nothing more to do.

Lemma 2.3. *The interval I_δ is open.*

Proof. If $t \in I_\delta$ and φ is a corresponding solution of (2.6), then the Ricci curvature of the associated metric ω_φ is equal to

$$t\,\omega_\varphi \,+\, \left((\mu - t) + \frac{\delta(1-\beta)\lambda}{\delta + ||S||_0^2}\right)\omega_0 \,+\, \delta(1-\beta)\,\frac{DS \wedge \overline{DS}}{(\delta + ||S||_0^2)^2}.$$

So $\mathrm{Ric}(\omega_\varphi) > t\,\omega_\varphi$. By the well-known Bochner identity, the first non-zero eigenvalue of ω_φ is strictly bigger than t. It implies that the linearization $\Delta_t + t$ of (2.6) at φ is invertible, where Δ_t is the Laplacian of ω_φ. By the Implicit Function Theorem, (2.6) is solvable for any t' close to t, so I_δ is open. □

Therefore, we only need to prove that I_δ is closed. This is amount to a priori estimates for any derivatives of the solutions of (2.6). As usual, by using known techniques in deriving higher order estimates, we need to bound only $J_{\omega_0}(\varphi)$ for any solution φ of (2.6) (cf. [Ti97], [Ti98]). The following arguments are identical to those for proving that the properness of $\mathbf{F}_{\omega_0,1}$ implies the existence of the Kähler-Einstein metrics in Theorem 1.6 of [Ti97].

We introduce

$$\mathbf{F}_{\delta,t}(\varphi) \,=\, \mathbf{J}_{\omega_0}(\varphi) \,-\, \frac{1}{V}\int_M \varphi\,\omega_0^n \,-\, \frac{1}{t}\,\log\left(\frac{1}{V}\int_M e^{h_\delta - t\varphi}\,\omega_0^n\right). \tag{2.7}$$

This is the Lagrangian of (2.6).

Lemma 2.4. *There is a constant C independent of t satisfying: For any smooth family of φ_s $(s \in [0,t])$ such that $\varphi = \varphi_t$ and φ_s solves* (2.6) *with parameter s, we have*

$$\mathbf{F}_{\delta,t}(\varphi) \,\le\, C.$$

Proof. First we observe

$$\mathbf{F}_{\delta,s}(\varphi_s) \,=\, \mathbf{J}_{\omega_0}(\varphi_s) \,-\, \frac{1}{V}\int_M \varphi_s\,\omega_0^n. \tag{2.8}$$

So its derivative on s is given by

$$\frac{d}{ds}\,\mathbf{F}_{\delta,s}(\varphi_s) \,=\, \frac{1}{sV}\int_M \varphi_s\,(\omega_0 + \sqrt{-1}\,\partial\bar\partial\varphi_s)^n.$$

Here we have used the fact

$$\int_M (\dot\varphi_s \,+\, s\,\varphi_s)\,(\omega_0 + \sqrt{-1}\,\partial\bar\partial\varphi_s)^n \,=\, 0$$

This follows from differentiating (2.6) on s.

We will show that the derivative in (2.8) is bounded from above. Without loss of the generality, we may assume that $s \geq s_0 > 0$. Then we have

$$\mathrm{Ric}(\omega_{\varphi_s}) \,\geq\, s\,\omega_{\varphi_s} \,\geq\, s_0\,\omega_{\varphi_s},$$

and consequently, the Sobolev constant of ω_{φ_s} is uniformly bounded. By the standard Moser iteration, we have (cf. [Ti98])

$$-\inf_M \varphi_s \,\leq\, -\frac{1}{V}\int_M \varphi_s\,(\omega_0 + \sqrt{-1}\partial\bar\partial\varphi_s)^n \,+\, C'.$$

Since $\inf_M \varphi_s \,\leq\, 0$, we get

$$\frac{d}{ds}\,\mathbf{F}_{\delta,s}(\varphi_s) \,\leq\, s_0^{-1}C'.$$

The lemma follows from integration along s. □

Next we observe for any $t \leq \mu$

$$h_\delta \,=\, h_0 - (1-\beta)\log(\delta + ||S||_0^2) + c_\delta \,\leq\, h_0 - (1-\beta_t)\log||S||_0^2 + c_\delta.$$

Hence, by Corollary 2.2, we have

$$\mathbf{F}_{\delta,t}(\psi) \,\geq\, \epsilon\,\mathbf{J}_{\omega_0}(\psi) \,-\, C_\epsilon \,-\, \frac{c_\delta - a_\beta}{t}.$$

Since both c_δ and a_β are uniformly bounded, combined with Lemma 2.4, we conclude that $J_{\omega_0}(\varphi)$ is uniformly bounded for any solution φ of (2.6).[9] Thus we have proved

Theorem 2.5. *For any $\delta > 0$, (2.5) has a unique smooth solution φ_δ. Consequently, we have a Kähler metric $\omega_\delta = \omega_0 + \sqrt{-1}\,\partial\bar\partial\varphi_\delta$ with Ricci curvature greater than or equal to μ.*

Next we examine the limit of ω_δ or φ_δ as δ tends to 0. First we note that for the conic Kähler-Einstein metric ω with cone angle $2\pi\beta$ along D given above, there is a uniform constant $c = c(\omega)$ such that $\sup_M |\varphi_\delta| \,\leq\, c$. Using $\mathrm{Ric}(\omega_\delta) \,\geq\, \omega_\delta$ and the standard computations, we have

$$\Delta \log \mathrm{tr}_{\omega_\delta}(\omega_0) \,\geq\, -a\,\mathrm{tr}_{\omega_\delta}(\omega_0),$$

where Δ is the Laplacian of ω_δ and a is an upper bound of the bisectional curvature of ω_0. If we put

$$u \,=\, \mathrm{tr}_{\omega_\delta}(\omega_0) - (a+1)\,\varphi_\delta,$$

then it follows from the above

$$\Delta u \,\geq\, u - n - (a+1)\,c.$$

[9]Here we also used the fact that $J_{\omega_0}(\varphi)$ is automatically bounded for $t > 0$ sufficiently small.

Hence, we have

$$u \leq n + (a+1)\, c,$$

this implies

$$C^{-1}\, \omega_0 \leq \omega_\delta,$$

where $C = n + 2(a+1)\, c$. Using the equation (2.6), we have

$$C^{-1}\, \omega_0 \leq \omega_\delta \leq C' \, (\delta + ||S||^2)^{-(1-\beta)}\, \omega_0, \tag{2.9}$$

where C' is a constant depending only on a and ω_0. Since $\beta > 0$, the above estimate on $\omega_\delta = \omega_0 + \sqrt{-1}\, \partial\bar\partial\varphi_\delta$ gives the uniform Hölder continuity of φ_δ. Furthermore, using the Calabi estimate for the 3rd derivatives and the standard regularity theory, we can prove (cf. [Ti98]): For any $l > 2$ and a compact subset $K \subset M\backslash D$, there is a uniform constant $C_{l,K}$ such that

$$||\varphi_\delta||_{C^l(K)} \leq C_{l,K}. \tag{2.10}$$

Then we can deduce from the above estimates:

Theorem 2.6. *The smooth Kähler metrics ω_δ converge to ω in the Gromov-Hausdorff topology on M and in the smooth topology outside D.*

Proof. It suffices to prove the first statement: ω_δ converge to ω in the Gromov-Hausdorff topology. Since ω_δ has Ricci curvature bounded from below by a fixed $\mu > 0$, by the Gromov Compactness Theorem, any sequence of (M, ω_δ) has a subsequence converging to a length space $(\bar M, \bar d)$ in the Gromov-Hausdorff topology. We only need to prove that any such a limit $(\bar M, \bar d)$ coincides with (M, ω). Without loss of generality, we may assume that (M, ω_δ) converge to $(\bar M, \bar d)$ in the Gromov-Hausdorff topology. By the estimates on derivatives in (2.10), $\bar M$ contains an open subset U which can be identified with $M\backslash D$, moreover, this identification $\iota : M\backslash D \mapsto U$ is an isometry between $(M\backslash D, \omega|_{M\backslash D})$ and $(U, \bar d|_U)$. On the other hand, since ω is a conic metric with angle $2\pi\beta \leq 2\pi$ along D, one can easily show by standard arguments that $M\backslash D$ is geodesically convex with respect to ω. Then it follows from (2.9) that M is the metric completion of $M\backslash D$ and ι extends to a Lipschtz map from (M, ω) onto $(\bar M, \bar d)$, still denoted by ι. In fact, the Lipschtz constant is 1.

We claim that ι is an isometry. This is equivalent to the following: For any p and q in $M\backslash D$,

$$d_\omega(p, q) = \bar d(\iota(p), \iota(q)).$$

It also follows from (2.9) that $\bar D = \iota(D)$ has Hausdorff measure 0 and is the Gromov-Hausdorff limit of D under the convergence of (M, ω_δ) to $(\bar M, \bar d)$. To prove the above claim, we only need to prove: For any $\bar p, \bar q \in \bar M\backslash\bar D$, there is a minimizing geodesic $\gamma \subset \bar M\backslash\bar D$ joining $\bar p$ to $\bar q$. Its proof is based on a relative volume comparison estimate due to Gromov ([Gr97], p 523, (B)). [10] We will prove it by contradiction. If no such a geodesic joins $\bar p$ to $\bar q$, then

$$\bar d(\bar p, \bar q) < d_\omega(p, q),$$

[10] I am indebted to Jian Song for this reference. He seems to be the first of applying such an estimate to studying the convergence problem in Kähler geometry.

where $\bar{p} = \iota(p)$ and $\bar{q} = \iota(q)$. Then there is a $r > 0$ satisfying:

(1) $B_r(\bar{p}, \bar{d}) \cap \bar{D} = \emptyset$ and $B_r(\bar{q}, \bar{d}) \cap \bar{D} = \emptyset$, where $B_r(\cdot, \bar{d})$ denotes a geodesic ball in $(\bar{M}, \bar{d})$;

(2) $\bar{d}(\bar{x}, \bar{y}) < d_\omega(x, y)$, where $\bar{x} = \iota(x) \in B_r(\bar{p}, \bar{d})$ and $\bar{y} = \iota(y) \in B_r(\bar{q}, \bar{d})$.

It follows from (1) and (2) that any minimizing geodesic γ from $\bar{x}$ to $\bar{y}$ intersects with $\bar{D}$. By choosing r sufficiently small, we may have

$$B_r(\bar{p}, \bar{d}) = \iota(B_r(p, \omega)) \quad \text{and} \quad B_r(\bar{q}, \bar{d}) = \iota(B_r(q, \omega)).$$

Choose a small tubular neighborhood T of D in M whose closure is disjoint from both $B_r(p, \omega)$ and $B_r(q, \omega)$. It is easy to see that T can be chosen to have the volume of ∂T as small as we want. Now we choose $p_\delta, q_\delta \in M$ and neighborhood T_δ of D with respect to ω_δ such that in the Gromov-Haudorff convergence,

$$\lim_{\delta \to 0+} p_\delta \,=\, \bar{p}\,, \quad \lim_{\delta \to 0+} q_\delta \,=\, \bar{q}\,, \quad \lim_{\delta \to 0+} T_\delta \,=\, \iota(T)\,.$$

It follows

$$\lim_{\delta \to 0+} Vol(\partial T_\delta, \omega_\delta) \,=\, Vol(\partial T, \omega).$$

Also, for δ sufficiently small, $B_r(p_\delta, \omega_\delta)$, $B_r(q_\delta, \omega_\delta)$ and T_δ are mutually disjoint. Clearly, any minimizing geodesic γ_δ from any $w \in B_r(p_\delta, \omega_\delta)$ to $z \in B_r(q_\delta, \omega_\delta)$ intersects with T_δ, so by Gromov's estimate ([Gr97], p523, (B)),

$$c\, r^{2n} \,\le\, Vol(B_r(q_\delta, \omega_\delta), \omega_\delta) \,\le\, C\, Vol(\partial T_\delta, \omega_\delta),$$

where c depends only on β and C depends only on μ, n, r. This leads to a contradiction because $Vol(\partial T_\delta, \omega_\delta)$ converge to $Vol(\partial T, \omega)$ which can be made as small as we want. Thus, ι is an isometry and our theorem is proved.

$\square$

Finally, we exam the limit of $\rho_{\omega_\delta, \ell}$ for any $\ell > 0$.

First we associate a Hermitian norm $|| \cdot ||_0^2$ to ω_0: For any section σ of K_M^{-1}, in local coordinates $z_1, \cdots, z_n$, we can write

$$\sigma \,=\, f \frac{\partial}{\partial z_1} \wedge \cdots \wedge \frac{\partial}{\partial z_n},$$

then

$$||\sigma||_0^2 \,=\, e^{h_0} \det(g_{i\bar{j}})\, |f|^2,$$

where $\omega_0 \,=\, \sqrt{-1}\, g_{i\bar{j}}\, dz_i \wedge d\bar{z}_j$. The curvature form of $|| \cdot ||_0^2$ is ω_0.

Similarly, we can associate a Hermitian norm $|| \cdot ||_\delta^2$ whose curvature is ω_δ. Using (2.5), we see

$$|| \cdot ||_\delta^2 \,=\, e^{c'_\delta - \varphi_\delta}\, || \cdot ||_0^2,$$

where c'_δ satisfies

$$\int_M \left(e^{h_0 - \varphi_\delta + c'_\delta} \,-\, 1 \right) \omega_0^n \,=\, 0.$$

Then as $\delta \to 0$, Hermitian norms $||\cdot||_\delta^2$ converge to the Hermitian norm on K_M^{-1}:[11]

$$||\cdot||^2 \,=\, e^{c'-\varphi}\,||\cdot||_0^2,$$

where φ is the solution of (2.1) and c' satisfies:

$$\int_M \left(e^{h_0-\varphi+c'} \,-\, 1\right)\omega_0^n \,=\, 0.$$

If we denote by $||\cdot||_\beta^2$ the Hermitian norm on K_M^{-1} defined by the volume form of ω_φ, then

$$||\cdot||^2 \,=\, e^{c'-a_\beta}\,||S||_\beta^{2(1-\beta)}\,||\cdot||_\beta^2.$$

Theorem 2.7. *For any $\ell > 0$, let $< \cdot,\cdot >_\delta$ be the inner product on $H^0(M, K_M^{-\ell})$ induced by ω_δ and the Hermitian metric $||\cdot||_\delta^2$ on K_M^{-1}. Then as δ tends to 0, $< \cdot,\cdot >_\delta$ converge to the corresponding inner product by the Hermitian metric $||\cdot||^2$ and ω. In particular, when ℓ is sufficiently large, $\rho_{\omega_\delta,\ell}$ converge to $\rho_{\omega,\ell}$.*

Proof. We have seen above that φ_δ converges to φ in a Hölder continuous norm. It follows that the volume forms ω_δ^n converge to ω^n in the L^p-topology for any given $p \in (1,(1-\beta)^{-1})$ and the Hermitian metrics $||\cdot||_\delta^2$ converge to $||\cdot||^2$. Since the inner products $< \cdot,\cdot >_\delta$ are defined by these Hermitian metrics and volumes forms, the theorem follows easily. □

3 An extension of Cheeger-Colding-Tian

In this section, we show a compactness theorem on conic Kähler-Einstein metrics. This theorem, coupled with the smooth convergence result in the next section, extends a result of Cheeger-Colding-Tian [CCT95] on smooth Kähler-Einstein metrics. In fact, our proof makes use of results in [CCT95] with injection of some new technical ingredients.

Let ω_i be a sequence of conic Kähler-Einstein metrics with cone angle $2\pi\beta_i$ along D, so we have

$$\mathrm{Ric}(\omega_i) \,=\, \mu_i\omega_i \,+\, 2\pi(1-\beta_i)[D], \qquad \mu_i \,=\, 1-(1-\beta_i)\lambda.$$

We assume that $\beta_i \geq \epsilon > 0$ and $\lim \beta_i = \beta_\infty$, it follows $\lim \mu_i = \mu_\infty > 0$.

For each ω_i, we use Theorem 2.6 to get a smooth Kähler metric $\tilde\omega_i$ satisfying:

A1. Its Kähler class $[\tilde\omega_i] \,=\, 2\pi c_1(M)$;

A2. Its Ricci curvature $\mathrm{Ric}(\tilde\omega_i) \,\geq\, \mu_i$;

A3. The Gromov-Hausdorff distance $d_{GH}(\omega_i,\tilde\omega_i)$ is less that $1/i$.

By the Gromov compactness theorem, a subsequence of $(M,\tilde\omega_i)$ converges to a metric space (M_∞, d_∞) in the Gromov-Hausdorff topology. For simplicity,

[11] For simplicity of notations, we do not make explicit the dependence of $||\cdot||_\delta$ and $||\cdot||$ on μ.

we may assume that $(M, \tilde{\omega}_i)$ converges to (M_∞, d_∞). It follows from (3) above that (M, ω_i) also converges to (M_∞, d_∞) in the Gromov-Hausdorff topology.

Theorem 3.1. *There is a closed subset $\mathcal{S} \subset M_\infty$ of Hausdorff codimension at least 2 such that $M_\infty \backslash \mathcal{S}$ is a smooth Kähler manifold and d_∞ is induced by a Kähler-Einstein metric ω_∞ outside $\mathcal{S}$, that is,*

$$\mathrm{Ric}(\omega_\infty) = \mu_\infty \, \omega_\infty \quad \text{ on } \; M_\infty \backslash \mathcal{S}.$$

If $\beta_\infty < 1$, then ω_i converges to ω_∞ in the C^∞-topology outside $\mathcal{S}$. Moreover, if $\beta_\infty = 1$, the set $\mathcal{S}$ is of codimension at least 4 and ω_∞ extends to a smooth Kähler metric on $M_\infty \backslash \mathcal{S}$.

This theorem is essentially due to Z.L. Zhang and myself [TZ12]. In this joint work, we develop a regularity theory for conic Einstein metrics which generalizes the work of Cheeger-Colding and Cheeger-Colding-Tian. Here, for completion and convenience, we give an alternative proof by using the approximations from last section.

Proof. Using the fact that (M_∞, d_∞) is the Gromov-Hausdorff limit of $(M, \tilde{\omega}_i)$, we can deduce from [CC95] the existence of tangent cones at every $x \in M_\infty$. More precisely, given any $x \in M_\infty$, for any $r_i \mapsto 0$, by taking a subsequence if necessary, $(M_\infty, r_i^{-2} d_\infty, x)$ converges to a tangent cone $\mathcal{C}_x$ at x. Define $\mathcal{R}$ to be the set of all points x in M_∞ such that some tangent cone $\mathcal{C}_x$ is isometric to $\mathbb{R}^{2n}$.

First we prove that $\mathcal{R}$ is open. If $\beta_\infty = 1$, then $\lim \mu_i = 1$. Since

$$[\tilde{\omega}_i] = 2\pi \, c_1(M) \quad \text{and} \quad \mathrm{Ric}(\tilde{\omega}_i) \geq \mu_i \, \tilde{\omega}_i,$$

by the arguments in the proof of Theorem 6.2 in [TW11], one can show that $(M, \tilde{\omega}_i)$ is a sequence of almost Kähler-Einstein metrics in the sense of [TW11]. Then it follows from Theorem 2 in [TW11] that M_∞ is smooth outside a closed subset $\mathcal{S}$ of codimension at least 4 and d_∞ is induced by a smooth Kähler-Einstein metric ω_∞ on $M_\infty \backslash \mathcal{S}$.

Now assume that $\beta_\infty < 1$. Note that (M, ω_i) also converge to (M_∞, d_∞) in the Gromov-Hausdorff topology. Let $\{x_i\}$ be a sequence of points in M which converge to $x \in \mathcal{R}$ during (M, ω_i)'s converging to (M_∞, d_∞). Since $x \in \mathcal{R}$, there is a tangent cone $\mathcal{C}_x$ of (M_∞, d_∞) at x which is isometric to $\mathbb{R}^{2n}$. It follows that for any $\epsilon > 0$, there is a $r = r(\epsilon)$ such that

$$\frac{Vol(B_r(x, d_\infty))}{r^{2n}} \geq c(n) - \epsilon,$$

where $c(n)$ denotes the volume of the unit ball in $\mathbb{R}^{2n}$. On the other hand, if $y_i \in D$, then by the Bishop-Gromov volume comparison, for any $\tilde{r} > 0$, we have

$$\frac{Vol(B_{\tilde{r}}(y_i, \omega_i))}{\tilde{r}^{2n}} \leq c(n) \, \beta_i.$$

It also follows from the Boshop-Gromov volume comparison that there is an $N = N(\epsilon)$ such that for any small $\bar{r} \in (0, r/N)$ and $y_i \in B_{\bar{r}}(x_i, \omega_i)$, we have

$$1 - \epsilon \le \frac{Vol(B_r(y_i, \omega_i))}{Vol(B_r(x_i, \omega_i))} \le 1 + \epsilon.$$

Now we claim that if $\bar{r} = r/N$, we have $B_{\bar{r}}(x_i, \omega_i) \cap D = \emptyset$. If this claim is false, say $y_i \in B_{\bar{r}}(x_i, \omega_i) \cap D$, then for i sufficiently large, we can deduce from the above and a result of Colding [Co94] on the volume convergence in the Gromov-Hausdorff topology

$$c(n) - 2\epsilon \le \frac{Vol(B_r(x_i, \omega_i))}{r^{2n}} \le (1 + \epsilon) \frac{Vol(B_r(y_i, \omega_i))}{r^{2n}} \le c(n)(1 + \epsilon)\, \beta_i.$$

Then we get a contradiction if ϵ is chosen sufficiently small. The claim is proved.

Since $B_{\bar{r}}(x_i, \omega_i)$ is contained in the smooth part of (M, ω_i) and its volume is sufficiently close to that of an Euclidean ball, the curvature of ω_i is uniformly bounded on the half ball $B_{3\bar{r}/4}(x_i, \omega_i)$ (cf. [An90]). It follows that ω_i restricted to $B_{\bar{r}/2}(x_i, \omega_i)$ converge to a smooth Kähler-Einstein metric on $B_{\bar{r}/2}(x, d_\infty)$ and $B_{\bar{r}/2}(x, d_\infty) \subset \mathcal{R}$. So $\mathcal{R}$ is open and d_∞ restricted to $\mathcal{R}$ is induced by a smooth Kähler-Einstein metric ω_∞.

The rest of the proof is standard in view of [CCT95].

Let $\mathcal{S}_k$ $(k = 0, 1, \cdots, 2n-1)$ denote the subset of M_∞ consisting of points for which no tangent cone splits off a factor, $\mathbb{R}^{k+1}$, isometrically. Clearly, $\mathcal{S}_0 \subset \mathcal{S}_1 \subset \cdots \subset \mathcal{S}_{2n-1}$. It is proved by Cheeger-Colding that $\mathcal{S}_{2n-1} = \emptyset$, $\dim \mathcal{S}_k \le k$ and $\mathcal{S} = \mathcal{S}_{2n-2}$. Moreover, if $\beta_\infty = 1$, it follows from [TW11] that $\mathcal{S} = \mathcal{S}_{2n-4}$. Then we have proved this theorem. $\square$

Using the same arguments in [CCT95], one can show:

Theorem 3.2. *Let $\mathcal{C}_x$ be a tangent cone of M_∞ at $x \in \mathcal{S}$, then we have*

C1. *Each $\mathcal{C}_x$ is regular outside a closed subcone $\mathcal{S}_x$ of complex codimension at least* 1. *Such a $\mathcal{S}_x$ is the singular set of $\mathcal{C}_x$;*

C2. *$\mathcal{C}_x = \mathbb{C}^k \times \mathcal{C}'_x$, in particular, $\mathcal{S}_{2k+1} = \mathcal{S}_{2k}$. We will denote o the vertex of $\mathcal{C}_x$;*

C3. *There is a natural Kähler Ricci-flat metric g_x whose Kähler form ω_x is $\sqrt{-1}\,\partial\bar\partial\rho_x^2$ on $\mathcal{C}_x \backslash \mathcal{S}_x$ which is also a cone metric, where ρ_x denotes the distance function from the vertex of $\mathcal{C}_x$;*

C4. *For any $x \in \mathcal{S}_{2n-2}$, $\mathcal{C}_x = \mathbb{C}^{n-1} \times \mathcal{C}'_x$, where $\mathcal{C}'_x$ is a 2-dimensional flat cone of angle $2\pi\bar\mu$ such that $0 < \bar\beta_\infty \le \bar\mu \le \beta_\infty$ and $(1 - \bar\mu) = m(1 - \beta_\infty)$ for some integer $m \ge 1$, where $\bar\beta_\infty$ depends only on β_∞.*

In fact, **C1**, **C2** and **C3** follow directly from results in [CCT95]. The proof of **C4** uses the slicing argument in [CCT95] for proving that $\mathcal{S}_{2n-2} = \emptyset$ in the case of smooth Kähler-Einstein metrics. In our new case, the conic singularity of ω_i along D may contribute a term close to $2\pi(1 - \beta_i)$ in the slicing argument,

this is how we can conclude that $\mathcal{C}'_x$ is a 2-dimensional flat cone of angle $2\pi\bar{\mu}$. The bounds on $\bar{\mu}$ follow from the Bishop-Gromov volume comparison. Note that $\bar{\beta}_\infty$ depends only on the diameter and volume of M_∞. Hence, there are only finitely many of such $\bar{\mu}$.

Next we state a corollary of Theorem 2.6:

Lemma 3.3. *There is a uniform bound on the Sobolev constants of (M, ω_i), that is, there is a constant C such that for any $f \in C^1(M, \mathbb{R})$,*

$$\left(\int_M |f|^{\frac{2n}{n-1}}\right)^{\frac{n-1}{n}} \omega_i^n \;\le\; C\int_M (|df|^2_{\omega_i} \,+\, |f|^2)\,\omega_i^n. \tag{3.1}$$

Proof. By Theorem 2.6, for any i, there is a sequence of smooth Kähler metrics $\omega_{i,\delta}$ converging to ω_i in the Gromov-Hausdorff topology and $\mathrm{Ric}(\omega_{i,\delta}) \,\ge\, \mu_i\,\omega_{i,\delta}$. Since the volume of $\omega_{i,\delta}$ is fixed, it is well-known that (3.1) holds uniformly for $\omega_{i,\delta}$. Then the lemma follows by taking $\delta \to 0$. □

4 Smooth convergence

We will adopt the notations from last section, e.g., ω_i is a conic Kähler-Einstein metric on M with angle $2\pi\beta_i$ along D as before. The main result of this section is to show that ω_i converge to ω_∞ outside a close subset of codimension at least 2. This is crucial for our establishing the partial C^0-estimate for conic Kähler-Einstein metrics as well as finishing the proof Theorem 1.1. This is related to the limit of D when (M, ω_i) converges to (M_∞, d_∞). If $\beta_\infty < 1$, the limit of D is in the singular set $\mathcal{S}$ of M_∞ since ω_i converge to ω_∞ in the C^∞-topology outside $\mathcal{S}$ as shown in last section. The difficulty lies in the case when $\beta_\infty = 1$. By [TW11], $\mathcal{S}$ is of codimension at least 4, so M_∞ is actually smooth outside a closed subset of codimension 4. Related results for smooth Kähler-Einstein metrics were proved before (cf. [CCT95], [Ch03]). However, a priori, it is not even clear if ω_i converge to ω_∞ in a stronger topology on any open subset of $M_\infty \backslash \mathcal{S}$. The original arguments in [CCT95] rely on an argument in [An90] which works only for smooth metrics. It fails for conic Kähler-Einstein metrics. So we need to have a new approach. In the course of proving our main result in this section, we also exam the limit of D in M_∞.

First we describe a general and important construction: Given any conic metric ω with cone angle $2\pi\beta$ along D, its determinant gives a Hermitian metric $\tilde{H}$ on K_M^{-1} outside D. For simplicity, we will also denote by $\tilde{H}$ the induced Hermitian metric on $K_M^{-\ell}$ for any $\ell > 0$. However, $\tilde{H}$ is singular along D, more precisely, if S is a defining section of D, then it is of the order $||S||_0^{-2(1-\beta)}$ along D, where $||\cdot||_0$ is a fixed Hermitian norm. This implies that $\tilde{H}(S,S)^{\frac{1-\beta}{\mu}}\tilde{H}$ is bounded along D, where $\mu = 1-(1-\beta)\lambda$. On the other hand, there is a unique f such that as currents,

$$\mathrm{Ric}(\omega) \,=\, \mu\,\omega \,+\, 2\pi(1-\beta)\,[D] \,+\, \sqrt{-1}\,\partial\bar{\partial}h,$$

where f is normalized by

$$\int_M \left(e^h - 1\right) \omega^n \;=\; 0.$$

Note that h is Hölder continuous. Put

$$H_\omega(\cdot,\cdot) \;=\; e^{\frac{h}{\mu}}\, \tilde{H}(S,S)^{\frac{1-\beta}{\mu}}\, \tilde{H}(\cdot,\cdot),$$

then as a current, the curvature of H_ω is equal to

$$\mathrm{Ric}(\omega) \;-\; \frac{1-\beta}{\mu}\sqrt{-1}\,\partial\bar\partial \log \tilde{H}(S,S) \;-\; \frac{\sqrt{-1}}{\mu}\,\partial\bar\partial h \;=\; \omega.$$

Also we normalize H_ω by scaling S such that

$$\int_M H_\omega(S,S)\,\omega^n \;=\; \int_M e^{\frac{\lambda h}{\mu}}\, \tilde{H}(S,S)^{\frac{1}{\mu}}\,\omega^n \;=\; 1.$$

Such a Hermitian metric H_ω is uniquely determined by ω and D and called the associated Hermitian metric of ω. If ω is conic Kähler-Einstein, its associated metric H_ω is determined by the volume form ω^n, e.g., in local holomorphic coordinates $z_1, \cdots, z_n$, write

$$\omega \;=\; \sqrt{-1}\, g_{i\bar{j}}\, dz_i \wedge d\bar{z}_j \quad \text{and} \quad S \;=\; f\left(\frac{\partial}{\partial z_1} \wedge \cdots \wedge \frac{\partial}{\partial z_n}\right)^{\lambda},$$

then H_ω is represented by

$$\det(g_{i\bar{j}})^{\frac{1}{\mu}}\, |f|^{\frac{2(1-\beta)}{\mu}}.$$

In particular, it implies that for any $\sigma \in H^0(M, K_M^{-\ell})$, $H_\omega(\sigma,\sigma)$ is bounded along D.

Now we recall some identities for pluri-anti-canonical sections.

Lemma 4.1. *Let ω_i be as above and H_i be the associated Hermitian metric on K_M^{-1}. Then for any $\sigma \in H^0(M, K_M^{-\ell})$, we have (in the sense of distribution)*

$$\Delta_i ||\sigma||_i^2 \;=\; ||\nabla\sigma||_i^2 \;-\; n\ell\, ||\sigma||_i^2 \tag{4.1}$$

and

$$\Delta_i ||\nabla\sigma||_i^2 \;=\; ||\nabla^2\sigma||_i^2 \;-\; ((n+2)\,\ell \;-\; \mu_i)\, ||\nabla\sigma||_i^2, \tag{4.2}$$

where $||\cdot||_i$ denotes the Hermitian norm on $K_M^{-\ell}$ induced by $H_i = H_{\omega_i}$, ∇ denotes the covariant derivative of H_i and Δ_i denotes the Laplacian of ω_i.

Proof. On $M\backslash D$, both (4.1) and (4.2) were already derived in [Ti90] by direct computations. Since $||\sigma||_i^2$ is bounded, (4.1) holds on M.

By a direct computation in local coordinates, one can also show that $||\nabla\sigma||_i^2$ is bounded along D, so (4.2) also holds. $\square$

Applying the standard Moser iteration to (4.1) and (4.2) and using Lemma 3.3, we obtain

Corollary 4.2. *There is a uniform constant C such that for any $\sigma \in H^0(M, K_M^{-\ell})$, we have*

$$\sup_M \Big(||\sigma||_i \,+\, \ell^{-\frac{1}{2}}\, ||\nabla \sigma||_i \Big) \,\le\, C\, \ell^{\frac{n}{2}} \, \left(\int_M ||\sigma||_i^2 \omega_i^n \right)^{\frac{1}{2}}. \tag{4.3}$$

If σ_i is a sequence in $H^0(M, K_M^{-\ell})$ satisfying:

$$\int_M ||\sigma_i||_i^2\, \omega_i^n \,=\, 1,$$

then by Corollary 4.2, $||\sigma_i||_i$ and their derivative are uniformly bounded. It implies that $||\sigma_i||_i$ are uniformly continuous. Hence, by taking a subsequence if necessary, we may assume $||\sigma_i||_i$ converge to a Lipschtz function F_∞ as i tends to ∞, moreover, we have

$$\int_{M_\infty} F_\infty^2\, \omega_\infty^n \,=\, 1.$$

In particular, F_∞ is non-zero.

Now we assume $\sigma_i = a_i S$, where a_i are constants and S is a defining section of D. Then $||\sigma_i||_i(x) \,=\, 0$ if and only if $x \in D$. If $F_\infty(x) \neq 0$ for some $x \in M_\infty \backslash \mathcal{S}$, then for a sufficiently small $r > 0$, we have

$$2\, F_\infty(y) \,\ge\, F_\infty(x) \,>\, 0, \quad \forall\, y \in B_r(x, \omega_\infty).$$

This is because F_∞ is continuous. This implies

$$B_r(x, \omega_\infty) \subset M_\infty \backslash \mathcal{S}.$$

Since $||\sigma_i||_i$ converge to F_∞ uniformly, for i sufficiently large, $||\sigma_i||_i \,>\, 0$ on those geodesic balls $B_r(x_i, \omega_i)$ of (M, ω_i) which converge to $B_r(x, \omega_\infty)$ in the Gromov-Hausdorff topology. It follows that $B_r(x_i, \omega_i) \subset M \backslash D$, that is, each $B_r(x_i, \omega_i)$ lies in the smooth part of (M, ω_i). On the other hand, since x is a smooth point of M_∞, by choosing smaller r, we can make the volume of $B_r(x_i, \omega_i)$ sufficiently close to that of corresponding Euclidean ball, then as one argued in [CCT95] by a result of [An90], ω_i restricted to $B_r(x_i, \omega_i)$ converge to ω_∞ on any compact subset of $B_r(x, \omega_\infty)$ in the C^∞-topology. Thus, ω_i converge to ω_∞ in the C^∞-topology on the non-empty open subset $M_\infty \backslash F_\infty^{-1}(0) \cup \mathcal{S}$.

Next we want to show that $F_\infty^{-1}(0)$ does not contain any open subset, or equivalently, $M_\infty \backslash F_\infty^{-1}(0)$ is an open-dense subset in M_∞. We prove it by contradiction. If it is false, say $U \subset F^{-1}(0)$ is open, using the fact that $||\sigma_i||_i$ is uniformly bounded from above, we have

$$\lim_{i \to \infty} \int_M \log(\frac{1}{i} + ||\sigma_i||_i^2)\, \omega_i^n \,=\, -\infty. \tag{4.4}$$

By a direct computation, we have

$$\omega_i + \sqrt{-1}\partial\bar\partial \log(\frac{1}{i} + ||\sigma_i||_i^2) = \frac{\omega_i}{1 + i\,||\sigma_i||_i^2} + \frac{i\, D\sigma_i \wedge \overline{D\sigma_i}}{(1 + i\,||\sigma_i||_i^2)^2} \ge 0.$$

It implies

$$\Delta_i \log(\frac{1}{i} + ||\sigma_i||_i^2) \ge -n.$$

Using the Sobolev inequality in Lemma 3.3 and the Moser iteration, we can deduce

$$\sup_M \log(\frac{1}{i} + ||\sigma_i||_i^2) \le C\left(1 + \int_M \log(\frac{1}{i} + ||\sigma_i||_i^2)\,\omega_i^n\right),$$

where C is a uniform constant. By (4.4),

$$\lim_{i\to\infty} \sup_M \log(\frac{1}{i} + ||\sigma_i||_i^2) = -\infty.$$

However, since the L^2-norm of $||\sigma_i||_i$ is equal to 1, there is a constant c independent of i such that

$$\sup_M \log(\frac{1}{i} + ||\sigma_i||_i^2) \ge -c.$$

This leads to a contradiction. Therefore, $M_\infty \backslash F_\infty^{-1}(0)$ is dense.

By our definition of the metric H_i associated to ω_i, in local holomorphic coordinates $z_1, \cdots, z_n$ away from D, we have

$$||\sigma||_i^2 = \left((\det(g_{a\bar b}))^\lambda\, |w|^2\right)^{\frac{1}{\mu}}$$

where

$$\sigma = w\,\frac{\partial}{\partial z_1} \wedge \cdots \wedge \frac{\partial}{\partial z_1} \quad \text{and} \quad \omega_i = \sqrt{-1}\sum_{a,b=1}^n g_{a\bar b}\, dz_a \wedge d\bar z_b\,.$$

Since ω_i converge to ω_∞ in the C^∞-topology outside $\mathcal{S}$, it follows from the above that σ_i converge to a holomorphic section σ_∞ on $M\backslash F_\infty^{-1}(0) \cup \mathcal{S}$.[12] Clearly, σ_∞ is bounded with respect to the Hermitian metric associated to ω_∞, so it extends to a holomorphic section of $K_{M_\infty}^{-\lambda}$ on the regular part $M\backslash\mathcal{S}$.

Next we exam the limit of D under the convergence of (M, ω_i). Since $||\sigma_i||_i = 0$ on D, the limit of D must lie in D_∞, where D_∞ denotes the zero set of F_∞. We claim that the limit of D coincides with D_∞. If this is not true, there are $x \in D_\infty$ and $r > 0$ such that $B_{2r}(x, d_\infty) \cap D_\infty$ is disjoint from the limit of D. Choose $x_i \in M$ go to x as (M, ω_i) converge to (M_∞, d_∞), then for i sufficiently large, $B_r(x_i, \omega_i)$ is disjoint from D, so lies in the smooth part

[12] The singular set $\mathcal{S}$ may overlap with $F_\infty^{-1}(0)$ along a subset of complex codimension 1.

of (M,ω_i). The regularity theory in [CCT95] implies that $\mathcal{S}\cap B_r(x,d_\infty)$ is of complex codimension at least 2 and near a generic point $y\in B_r(x,d_\infty)\cap D_\infty$, σ_∞ is holomorphic and defines D_∞, moreover, the convergence of (M,ω_i) to (M_∞,d_∞) is in C^∞-topology and σ_i converge to σ_∞ near y , so σ_i must vanish somewhere in $B_r(x_i,\omega_i)$, a contradiction. This shows that the limit of D coincides with D_∞.

If $\beta_\infty = 1$, the singular set $\mathcal{S}$ is of complex dimension at least 2 and $\sigma_\infty \in H^0(M_\infty, K_{M_\infty}^{-\lambda})$ which consists of all holomorphic sections of $K_{M_\infty}^{-\lambda}$ on $M_\infty\backslash\mathcal{S}$. Then D_∞ is simply the divisor $\{\sigma_\infty = 0\}$.

Summarizing the above discussions, we have

Theorem 4.3. *Let (M_∞,ω_∞), $\mathcal{S}$ etc. be as in Theorem 3.1. Then (M,ω_i) converge to (M_∞,ω_∞) in the C^∞-topology outside $\bar{\mathcal{S}}\cup D_\infty$ for a closed subset $\bar{\mathcal{S}}$ of codimension at least 4 and D converges to D_∞ in the Gromov-Hausdorff topology. If $\beta_\infty < 1$, $\mathcal{S} = \bar{\mathcal{S}}\cup D_\infty$. If $\beta_\infty = 1$, $\mathcal{S} = \bar{\mathcal{S}}$ and D_∞ is a divisor of $K_{M_\infty}^{-\lambda}$.*[13]

5 Partial C^0-estimate

In this section, we prove Theorem 1.2. By our results on compactness of conic Kähler-Einstein metrics in last two sections, we need to prove only the following:

Theorem 5.1. *Let M be a Fano manifold M and D be a smooth divisor whose Poincare dual is $\lambda\, c_1(M)$. Let ω_i be a sequence of conic Kähler-Einstein metrics on M with conic angle $2\pi\beta_i$ along D satisfying:*

$$\lim \beta_i \;=\; \beta_\infty > 0 \quad \text{and} \quad 0 \,<\, (1-\beta_\infty)\,\lambda \,<\, 1.$$

We also assume that (M,ω_i) converge to a (possibly singular) conic Kähler-Einstein manifold (M_∞,ω_∞) as described in Theorem 4.3. Then there are uniform constants $c_k = c(k,n,\lambda,\beta_\infty) > 0$ for $k\ge 1$ and $\ell_a\to\infty$ such that for $\ell=\ell_a$,

$$\rho_{\omega_i,\ell} \;\ge\; c_\ell \;>\; 0. \tag{5.1}$$

For the readers' convenience, we recall the definition of $\rho_{\omega_i,\ell}$: Let H_i be the Hermitian metric on K_M^{-1} associated to ω_i, then for any orthonormal basis $\{S_\alpha\}_{0\le\alpha\le N}$ of $H^0(M,K_M^{-\ell})$ with respect to the inner product induced by H_i and ω_i, we have

$$\rho_{\omega_i,\ell}(x) \;=\; \sum_{\alpha=0}^{N} H_i(S_\alpha,S_\alpha)(x), \tag{5.2}$$

We have shown in last section that the defining sections σ_i of D normalized with respect to H_i converge to a holomorphic section σ_∞ of $K_{M_\infty}^{-\lambda}$ on $M\backslash\mathcal{S}$

[13] It follows from the partial C^0-estimate in the next section that the same holds even if $\beta_\infty < 1$.

satisfying: In any local coordinates $z_1, \cdots, z_n$ outside $\mathcal{S}$, we have

$$(\det(g_{a\bar{b}}))^{\lambda} \, |w|^2 \, < \, \infty \tag{5.3}$$

where

$$\sigma_\infty \, = \, w \, \frac{\partial}{\partial z_1} \wedge \cdots \wedge \frac{\partial}{\partial z_1} \quad \text{and} \quad \omega_\infty \, = \, \sqrt{-1} \, g_{a\bar{b}} \, dz_a \wedge d\bar{z}_b \, .$$

Define a Hermitian metric H_∞ on $K_{M_\infty}^{-1}$ on $M_\infty \backslash \mathcal{S}$ by

$$H_\infty \, = \, \tilde{H}_\infty(\sigma_\infty, \sigma_\infty)^{\frac{1-\beta}{\mu}} \, \tilde{H}_\infty. \tag{5.4}$$

Here $\tilde{H}_\infty$ denotes the Hermitian metric induced by the determinant of ω_∞. The following can be easily proved.

Lemma 5.2. *The Hermitian metrics H_i converge to H_∞ on $M_\infty \backslash \mathcal{S}$ in the C^∞-topology. Moreover, we have*

$$H_\infty(\sigma_\infty, \sigma_\infty) \, < \, \infty \quad \text{and} \quad \int_{M_\infty} H_\infty(\sigma_\infty, \sigma_\infty) \, \omega_\infty^n \, = \, 1.$$

Let us first specify the holomorphic sections of $K_{M_\infty}^{-\ell}$ we will use here. [14] By a holomorphic section of $K_{M_\infty}^{-\ell}$ on M_∞ ($\ell > 0$), we mean a holomorphic section σ of $K_{M_\infty}^{-\ell}$ on $M_\infty \backslash \mathcal{S}$ with $H_\infty(\sigma, \sigma)$ bounded.

We denote by $H^0(M_\infty, K_{M_\infty}^{-\ell})$ the space of all holomorphic sections of $K_{M_\infty}^{-\ell}$ on M. If M_∞ is smooth outside a closed subset of codimension 4, then it coincides with the definition we used in literature.

Lemma 5.3. *For any fixed $\ell \, > \, 0$, if $\{\tau_i\}$ is any sequence of $H^0(M, K_M^{-\ell})$ satisfying:*

$$\int_M H_i(\tau_i, \tau_i) \, \omega_i^n \, = \, 1,$$

then a subsequence of τ_i converges to a section τ_∞ in $H^0(M_\infty, K_{M_\infty}^{-\ell})$.

This follows from the estimate in Corollary 4.2 and standard arguments. It implies that $\rho_{\omega_i,\ell}$ are uniformly continuous, in particular, they converge to a continuous function on M_∞. This function is actually equal to $\rho_{\omega_\infty,\ell}$ as shown in the end of this section, but we do not need this to prove Theorem 5.1.

Thus, in order to prove Theorem 5.1, we only need to show that for a sequence of ℓ,

$$\inf_i \inf_x \rho_{\omega_i,\ell}(x) \, > \, 0. \tag{5.5}$$

[14]This is needed since M_∞ may have singularity along a subset $\mathcal{S}$ of complex codimension 1. However, we will prove later that M_∞ is actually smooth outside a subset of complex codimension at least 2.

Since $\rho_{\omega_i,\ell}$ are uniformly continuous and M_∞ is compact, it suffices to show that for any $x \in M_\infty$, there is an ℓ and sequence $x_i \in M$ such that $\lim x_i = x$ and

$$\inf_i \rho_{\omega_i,\ell}(x_i) > 0. \tag{5.6}$$

The following lemma provides the L^2-estimate for $\bar\partial$-operator on (M, ω_i). It can be proved by using the smooth approximations $\tilde\omega_i$ of ω_i with Ricci curvature bounded from below.

Lemma 5.4. *For any $\ell > 0$, if ζ is a (0,1)-form with values in $K_M^{-\ell}$ and $\bar\partial\zeta = 0$, there is a smooth section ϑ of $K_M^{-\ell}$ such that $\bar\partial\vartheta = \zeta$ and*

$$\int_M ||\vartheta||_i^2\, \omega_i^n \le \frac{1}{\ell + \mu} \int_M ||\zeta||_i^2\, \omega_i^n,$$

where $||\cdot||_i$ denotes the norm induced by H_i and ω_i.

We have seen that for any $r_j \mapsto 0$, by taking a subsequence if necessary, we have a tangent cone $\mathcal{C}_x$ of $(M_\infty, \omega_\infty)$ at x, which is the limit of $(M_\infty, r_j^{-2}\omega_\infty, x)$ in the Gromov-Hausdorff topology, satisfying:

$\mathbf{T}_1$. Each $\mathcal{C}_x$ is regular outside a closed subcone $\mathcal{S}_x$ of complex codimension at least 1. Such a $\mathcal{S}_x$ is the singular set of $\mathcal{C}_x$;

$\mathbf{T}_2$. There is an natural Kähler Ricci-flat metric g_x on $\mathcal{C}_x \backslash \mathcal{S}_x$ which is also a cone metric. Its Kähler form ω_x is equal to $\sqrt{-1}\,\partial\bar\partial\rho_x^2$ on the regular part of $\mathcal{C}_x$, where ρ_x denotes the distance function from the vertex of $\mathcal{C}_x$, denoted by x for simplicity.

We will denote by L_x the trivial bundle $\mathcal{C}_x \times \mathbb{C}$ over $\mathcal{C}_x$ equipped with the Hermitian metric $e^{-\rho_x^2}\,|\cdot|^2$. The curvature of this Hermitian metric is given by ω_x.

As before, we denote by $\mathcal{S}_k$ $(k = 0, 1, \cdots, 2n-1)$ the subset of M_∞ consisting of points for which no tangent cone splits off a factor, $\mathbb{R}^{k+1}$, isometrically. Clearly, $\mathcal{S}_0 \subset \mathcal{S}_1 \subset \cdots \subset \mathcal{S}_{2n-1}$. It is proved by Cheeger-Colding that $\mathcal{S}_{2n-1} = \emptyset$, $\dim \mathcal{S}_k \le k$ and $\mathcal{S} = \mathcal{S}_{2n-2}$.

The following lemma can be proved by using the slicing arguments in [CCT95] and the fact that $(M_\infty, \omega_\infty)$ is the limit of conic Kähler-Einstein metrics (M, ω_i) with cone angle along $2\pi\beta_i$ along D.

Lemma 5.5. *For any $x \in \mathcal{S}_{2n-2} \backslash \bigcup_{k<2n-2} \mathcal{S}_k$, if $\mathcal{C}_x = \mathbb{C}^{n-1} \times \mathcal{C}'_x$, then g_x is a product of the Euclidean metric on $\mathbb{C}^{n-1}$ with a flat conic metric on $\mathcal{C}'_x$, which is biholomorphic to $\mathbb{C}$, of angle $2\pi\mu_a$ $(a = 1, \cdots, l)$, where $\bar\mu = \mu_a$ is given as in Theorem 3.2. Moreover, for any $x \in \mathcal{S} \subset M_\infty$, if $\mathcal{S}_x$ is of complex codimension 1, then there is a closed subcone $\bar{\mathcal{S}}_x \subset \mathcal{S}_x$ of complex codimension at least 2 such that g_x is asymptotic to the product metric described above at any $y \in \mathcal{S}_x \backslash \bar{\mathcal{S}}_x$, i.e., a tangent cone of $(\mathcal{C}_x, g_x)$ at y is isometric to a product of the Euclidean metric on $\mathbb{C}^{n-1}$ with a conic metric on $\mathcal{C}'_x$ of angle $2\pi\mu_a < 2\pi$.*

Remark 5.6. *As we said after Theorem 3.2, by the volume comparison, we know $\bar\beta_\infty \le \mu_a \le \beta_\infty$ for some $\bar\beta_\infty$ depending only on the diameter and volume of $(M_\infty, \omega_\infty)$. However, in our proof, we may assume that $\beta_\infty \ge 1 - \lambda^{-1} + \epsilon$ for some $\epsilon > 0$, so $\bar\beta_\infty$ can be assume to be uniform. When $\beta_\infty = 1$, all $\mu_a = 1$, so there is only one. If $\beta_\infty < 1$, since $(1 - \mu_a) = m_a(1 - \beta_\infty)$ for some integer m_a, there is a bound on l as well. In fact, one should be able to prove that there is a uniform bound on l depending only on λ.*

Without loss of generality, in the following, for each j, we set k_j to be the integral part of r_j^{-2}.

Now we fix some notations: For any $\epsilon > 0$, we put

$$V(x;\epsilon) \,=\, \{\, y \,\in\, \mathcal{C}_x \,|\, y \,\in\, B_{\epsilon^{-1}}(0, g_x) \setminus \overline{B_\epsilon(0, g_x)},\ d(y, \mathcal{S}_x) \,>\, \epsilon \,\},$$

where $B_R(o, g_x)$ denotes the geodesic ball of $(\mathcal{C}_x, g_x)$ centered at the vertex and with radius R.

If $\mathcal{C}_x$ has isolated singularity, then $\mathcal{S}_x \,=\, \{o\}$ and

$$V(x;\epsilon) \,=\, \{\, y \,\in\, \mathcal{C}_x \,|\, y \,\in\, B_{\epsilon^{-1}}(0, g_x) \setminus \overline{B_\epsilon(0, g_x)} \,\}.$$

Let r_j^{-2} be the above sequence such that $(M_\infty, r_j^{-2}\omega_\infty, x)$ converges to $(\mathcal{C}_x, g_x, o)$. By [CCT95], for any $\epsilon > 0$, whenever i is sufficiently large, there are diffeomorphisms $\phi_j : V(x;\epsilon) \mapsto M_\infty \backslash \mathcal{S}$, where $\mathcal{S}$ is the singular set of M_∞, satisfying:

(1) $d(x, \phi_j(V(x;\epsilon))) \,<\, 10\epsilon r_j$ and $\phi_j(V(x;\epsilon)) \subset B_{(1+\epsilon^{-1})r_j}(x)$, where $B_R(x)$ the geodesic ball of $(M_\infty, \omega_\infty)$ with radius R and center at x;

(2) If g_∞ is the Kähler metric with the Kähler form ω_∞ on $M_\infty \backslash \mathcal{S}$, then

$$\lim_{j\to\infty} ||r_j^{-2}\phi_j^* g_\infty - g_x||_{C^6(V(x;\frac{\epsilon}{2}))} \,=\, 0, \tag{5.7}$$

where the norm is defined in terms of the metric g_x.

Lemma 5.7. *For any δ sufficiently small, there are a sufficiently large $\ell \,=\, k_j$ and an isomorphism ψ from the trivial bundle $\mathcal{C}_x \times \mathbb{C}$ onto $K_{M_\infty}^{-\ell}$ over $V(x;\epsilon)$ commuting with $\phi \,=\, \phi_j$ satisfying:*

$$||\psi(1)||^2 \,=\, e^{-\rho_x^2} \quad \text{and} \quad ||\nabla\psi||_{C^4(V(x;\epsilon))} \,\le\, \delta, \tag{5.8}$$

where $||\cdot||^2$ denotes the induced norm on $K_{M_\infty}^{-\ell}$ by ω_∞, ∇ denotes the covariant derivative with respect to the norms $||\cdot||^2$ and $e^{-\rho_x^2}|\cdot|^2$.

Proof. The arguments of its proof are pretty standard, so we just outline it. We cover $V(x;\epsilon)$ by finitely many geodesic balls $B_{s_\alpha}(y_\alpha)$ $(1 \le \alpha \le N)$ satisfying:

(i) The closure of each $B_{2s_\alpha}(y_\alpha)$ is strongly convex and contained in $\mathrm{Reg}(\mathcal{C}_x)$;.

(ii) The half balls $B_{s_\alpha/2}(y_\alpha)$ are mutually disjoint.

Now we choose $\ell \,=\, \ell_j$ sufficiently large and construct ψ.

First we construct $\tilde{\psi}_\alpha$ over each $B_{2s_\alpha}(y_\alpha)$. For any $y \in B_{2s_\alpha}(y_\alpha)$, let $\gamma_y \subset B_{2s_\alpha}(y_\alpha)$ be the unique minimizing geodesic from y_α to y. We define $\tilde{\psi}_\alpha$ as follows: First we define $\tilde{\psi}_\alpha(1) \in L|_{\phi(y_\alpha)}$ such that

$$||\tilde{\psi}_\alpha(1)||^2 \,=\, e^{-\rho_x^2(y_\alpha)},$$

where $L \,=\, K_{M_\infty}^{-\ell}$. Next, for any $y \in U_\alpha$, where $U_\alpha \,=\, B_{2s_\alpha}(y_\alpha)$, define

$$\tilde{\psi}_\alpha : \mathbb{C} \mapsto L|_y, \quad \tilde{\psi}_\alpha(a(y)) = \tau(\phi(y)),$$

where $a(y)$ is the parallel transport of 1 along γ_y with respect to the norm $e^{-\rho_x^2}\,|\cdot|^2$ and $\tau(\phi(y))$ is the parallel transport of $\psi(1)$ along $\phi \circ \gamma_y$ with respect to the norm $||\cdot||^2$.

Clearly, we have the first equation in (5.8). The estimates on derivatives can be done as follows: If $a : U_\alpha \mapsto U_\alpha \times \mathbb{C}$ and $\tau : U_\alpha \mapsto \phi^* L|_{U_\alpha}$ are two sections such that $\tilde{\psi}_\alpha(a) \,=\, \tau$, then we have the identity:

$$\nabla \tau \,=\, \nabla \tilde{\psi}_\alpha(a) \,+\, \tilde{\psi}_\alpha(\nabla a),$$

where ∇ denote the covariant derivatives with respect to the given norms on line bundles $\mathcal{C}_x \times \mathbb{C}$ and L. By the definition, one can easily see that $\nabla \tilde{\psi}_\alpha(y_\alpha) \equiv 0$. To estimate $\nabla \tilde{\psi}_\alpha$ at y, we differentiate along γ_y to get

$$\nabla_T \nabla_X \tau \,=\, \nabla_T(\nabla_X \tilde{\psi}_\alpha(a)) + \tilde{\psi}_\alpha(\nabla_T \nabla_X a),$$

where T is the unit tangent of γ_y and X is a vector field along γ_y with $[T, X] = 0$. Here we have used the fact that $\nabla_T \tilde{\psi}_\alpha = 0$ which follows from the definition. Using the curvature formula, we see that it is the same as

$$k\phi^*\omega_\infty(T, X)\, \tilde{\psi}_\alpha(a) \,=\, \nabla_T(\nabla_X \tilde{\psi}_\alpha(a)) \,+\, \omega_x(T, X)\, a.$$

Using the fact that ω_x is the limit of $k\phi^*\omega_\infty$, we can deduce from the above that $\nabla_T(\nabla_X \tilde{\psi}_\alpha(a))$ converges to 0 as i tends to ∞. Since $\nabla_X \tilde{\psi}_\alpha = 0$ at y_α, we see that $||\nabla \tilde{\psi}_\alpha||_{C^0(U_\alpha)}$ can be made sufficiently small. The higher derivatives can be bounded in a similar way.

Next we want to modify each $\tilde{\psi}_\alpha$. For any α, β, we set

$$\theta_{\alpha\gamma} \,=\, \tilde{\psi}_\alpha^{-1} \circ \tilde{\psi}_\gamma : U_\alpha \cap U_\gamma \mapsto S^1.$$

Clearly, we have

$$\theta_{\alpha\kappa} \,=\, \theta_{\alpha\gamma} \cdot \theta_{\gamma\kappa} \quad \text{on} \;\; U_\alpha \cap U_\gamma \cap U_\kappa,$$

so we have a closed cycle $\{\theta_{\alpha\gamma}\}$. By the derivative estimates on each $\tilde{\psi}_\alpha$, we know that each $\theta_{\alpha\gamma}$ is close to a constant. Therefore, we can modify $\tilde{\psi}_\alpha$'s such that each transition function $\theta_{\alpha\gamma}$ is a unit constant, that is, we can construct $\zeta_\alpha : U_\alpha \mapsto S^1$ such that if we replace each $\tilde{\psi}_\alpha$ by $\tilde{\psi}_\alpha \cdot \zeta_\alpha$, the corresponding transition functions are constant. Moreover we can dominate $||\nabla \zeta_\alpha||_{C^3}$ by the norm $||\nabla \tilde{\psi}_\alpha||_{C^3}$ (possibly) on a slightly larger ball.

The cycle $\{\theta_{\alpha\gamma}\}$ of constants gives rise to a flat bundle F, and we have constructed an isomorphism

$$\xi : F \mapsto K_{M_\infty}^{-\ell}$$

over an neighborhood of $\overline{V(x;\epsilon)}$ satisfying all the estimates in (5.8).

If we replace ℓ by $k\ell$, we get an analogous isomorphism

$$\xi : F^k \mapsto K_{M_\infty}^{-k\ell}.$$

Since the flat bundle F is given by a representation

$$\rho : \pi_1(V(x;\epsilon)) \mapsto S^1,$$

there is an k such that F^k is essentially trivial, i.e., the corresponding transition functions are in a neighborhood of the identity in S^1. Then we can further modify $\tilde{\psi}_\alpha$ such that $\theta_{\alpha\gamma} = 1$ for any α and γ. So we can get the required ψ by setting $\psi = \tilde{\psi}_\alpha$ on $V(x;\epsilon) \cap B_{s_\alpha}(x_\alpha)$.

In fact, one can show that either of the following conditions holds for :

(1) There is a tangent cone $\mathcal{C}_x$ of the form $\mathbb{C}^{n-1} \times \mathcal{C}'_x$ for a 2-dimensional flat cone $\mathcal{C}'_x$ of angle $2\pi\mu_a$, where μ_a are given in Lemma 5.5 for $a = 1, \cdots, l$;

(2) There is a closed subcone $\bar{\mathcal{S}}_x \subset \mathcal{S}_x$ of codimension at least 4 such that for every $y \in \mathcal{S}_x \backslash \bar{\mathcal{S}}_x$, any tangent cone $\mathcal{C}_y$ of $\mathcal{C}_x$ at y is of the form $\mathbb{C}^{n-1} \times \mathcal{C}'_y$ for a 2-dimensional flat cone $\mathcal{C}'_y$ of angle $2\pi\mu_a$, where μ_a are given in Lemma 5.5 for $a = 1, \cdots, l$. Moreover, $\mathcal{C}_x \backslash \bar{\mathcal{S}}_x$ has finite fundamental group of order $\nu \geq 1$.

Thus we just need to take ℓ to be a multiple of ν such that for $a = 1, \cdots, l$, $\ell\mu_a$ are sufficiently close to 1 modulo $\mathbb{Z}$ in the above construction of ψ. Since $\mu_a = 1 - m + m\beta_\infty$ for some integer, the second condition is the same as requiring that $\ell\beta_\infty$ are sufficiently close to 1 modulo $\mathbb{Z}$. $\square$

As for smooth Kähler-Einstein metrics, we will apply the L^2-estimate to proving (5.6), consequently, the partial C^0-estimate for conic Kähler-Einstein metrics. The method is standard and resembles the one we used for Del-Pezzo surfaces in [Ti89]. First we construct an approximated holomorphic section $\tilde{\tau}$ on M_∞, then one can perturb it into a holomorphic section τ by the L^2-estimate for $\bar\partial$-operators, finally, one uses the derivative estimate in Corollary 4.2 to conclude that $\tau(x) \neq 0$.

Let $\epsilon > 0$ and $\delta > 0$ be sufficiently small and be determined later. We fix ℓ to be the integral part of r^{-2} and $r = r_j$ for a sufficiently large j which may depend on ϵ and δ. Choose ϕ and ψ by Lemma 5.7, then there is a section $\tau = \psi(1)$ of $K_{M_\infty}^{-\ell}$ on $\phi(V(x;\epsilon))$ satisfying:

$$||\tau||^2 = e^{-\rho_x^2}.$$

By Lemma 5.7, for some uniform constant C, we have

$$||\bar\partial\tau|| \leq C\,\delta.$$

Now let us state a technical lemma.

Lemma 5.8. *For any $\bar\epsilon > 0$, there is a smooth function $\gamma_{\bar\epsilon}$ on $\mathcal{C}_x$ satisfying:*

(1) $\gamma_{\bar\epsilon}(y) = 1$ for any y with $d(y, \mathcal{S}_x) \geq \bar\epsilon$, where $d(\cdot,\cdot)$ is the distance of $(\mathcal{C}_x, g_x)$;

(2) $0 \leq \gamma_{\bar\epsilon} \leq 1$ and $\gamma_{\bar\epsilon}(y) = 0$ in an neighborhood of $\mathcal{S}_x$;

(3) $|\nabla\gamma_{\bar\epsilon}| \leq C$ for some constant $C = C(\bar\epsilon)$ and

$$\int_{B_{\bar\epsilon^{-1}}(o,g_x)} |\nabla\gamma_{\bar\epsilon}|^2\,\omega_x^n \leq \bar\epsilon.$$

Proof. This is rather standard and has been known to me for quite a while. This is based on the fact that the Poincare metric on a punctured disc has finite volume.

First we consider the simplest case that $\mathcal{S}_x = \mathbb{C}^{n-1}$, i.e., $\mathcal{C}_x$ is of the form $\mathbb{C}^{n-1} \times \mathcal{C}'_x$, where $\mathcal{C}'_x$ is biholomorphic to $\mathbb{C}$. Moreover, the cone metric g_x coincides with the standard cone metric

$$g_{\bar\beta} = \sum_{i=1}^{n-1} dz_i d\bar z_i + (d\rho^2 + \bar\beta^2\rho^2 d\theta^2),$$

where $z_1, \cdots, z_{n-1}$ are coordinates of $\mathbb{C}^{n-1}$ and $\bar\beta$ is one of μ_a given in Lemma 5.5. Clearly, $\rho = d(y, \mathcal{S}_x)$.

We denote by η a cut-off function: $\mathbb{R} \mapsto \mathbb{R}$ satisfying: $0 \leq \eta \leq 1$, $|\eta'(t)| \leq 1$ and

$$\eta(t) = 0 \text{ for } t > \log(-\log\delta^3) \text{ and } \eta(t) = 1 \text{ for } t < \log(-\log\delta).$$

Here $\delta < 1/3$ is to be determined. Now we define as follows: If $\rho(y) \geq \bar\epsilon/3$, put $\gamma_{\bar\epsilon}(y) = 1$ and if $\rho(y) < \bar\epsilon$

$$\gamma_{\bar\epsilon}(y) = \eta\left(\log\left(-\log\left(\frac{\rho(y)}{\bar\epsilon}\right)\right)\right).$$

Clearly, $\gamma_{\bar\epsilon}$ is a smooth function and we have

$$\gamma_{\bar\epsilon}(y) = 1 \text{ if } \rho(y) \geq \frac{\bar\epsilon}{3} \quad\text{and}\quad \gamma_{\bar\epsilon}(y) = 0 \text{ if } \rho(y) \leq \delta^3\bar\epsilon.$$

Furthermore, the support of $|\nabla\gamma_{\bar\epsilon}|(y) = 0$ is contained in the region where $\delta^3\bar\epsilon < \rho(y) < \delta\bar\epsilon$. In the region, we have

$$|\nabla\gamma_{\bar\epsilon}| \leq \frac{1}{\rho(-\log\frac{\rho}{\bar\epsilon})}.$$

It follows that

$$\int_{B_{\bar\epsilon^{-1}}(o,g_x)} |\nabla\gamma_{\bar\epsilon}|^2\,\omega_x^n \leq \frac{a_{n-1}}{\bar\epsilon^{2n-2}}\int_{\delta^3}^{\delta}\frac{dr}{r(-\log r)^2} \leq \frac{a_{n-1}}{\bar\epsilon^{2n-2}(-\log\delta)},$$

where a_{n-1} denotes the volume of the unit ball in $\mathbb{R}^{2n-2}$.

Now choose δ such that $a_{n-1} \le \bar{\epsilon}^{2n-1}(-\log \delta)$, then we have

$$\int_{B_{\bar{\epsilon}^{-1}}(o,g_x)} |\nabla \gamma_{\bar{\epsilon}}|^2 \le \bar{\epsilon}.$$

Clearly, we also have $|\nabla \gamma_{\bar{\epsilon}}| \le C$ for some $C = C(\bar{\epsilon})$.

In general, as we have shown in Section 3 by using the arguments of [CCT95], $\mathcal{S}_x$ is a union of $\mathcal{S}_x^0$ and $\bar{\mathcal{S}}_x$, where $\bar{\mathcal{S}}_x$ is a closed subcone and $\mathcal{S}_x^0$ is an open subcone of $\mathcal{S}_x$ which consists of all $y \in \mathcal{S}_x$ such that a tangent cone of $(\mathcal{C}_x, g_x)$ at y is isometric to $\mathbb{C}^{n-1} \times \mathcal{C}'_y$ with the standard metric $g_{\bar{\beta}}$, where $(1-\bar{\beta}) = k(1-\beta_\infty)$ for some integer k. Furthermore, $\bar{\mathcal{S}}_x$ is of complex codimension at least 2.

We expect the following:

$\mathbf{A}_1$ $\mathcal{C}_x$ is a variety near $\mathcal{S}_x^0$ and $\mathcal{S}_x^0$ is a subvariety.

This can be proved by establishing a local version of Theorem 5.9 and by using the simplest case of Lemma 5.8. We refer the readers to Remark 7.4 for more discussions. Now we explain how to derive Lemma 5.8 under Assumption $\mathbf{A}_1$. This is intended for illustrating the idea of the proof of Lemma 5.8 before getting too tedious arguments based on known techniques. We will complete the proof of Lemma 5.8 by using an analogous, but weaker, version of Assumption $\mathbf{A}_1$ in Appendix.

Clearly, $\mathbf{A}_1$ implies the following;

$\mathbf{A}'_1$. $\mathcal{S}_x$ can be written as a union of two subcones $\mathcal{S}_{x,1}$ and $\mathcal{S}_{x,2}$ such that $\mathcal{S}_{x,2}$ is a closed subcone of complex codimension at least 2 and $\mathcal{C}_x$ is smooth near $\mathcal{S}_{x,1}$ which is a smooth divisor.

For any small $\epsilon_0 > 0$, since $\mathcal{S}_{x,2}$ has vanishing Hausdorff measure of dimension strictly bigger than $2n-4$, we can find a finite cover of $\mathcal{S}_{x,2} \cap B_{\bar{\epsilon}^{-1}}(x, g_x)$ by balls $B_{r_a}(y_a, g_x)$ $(a = 1, \cdots, l)$ satisfying:

(i) $y_a \in \mathcal{S}_{x,2}$ and $2r_a \le \epsilon_0$;

(ii) $B_{r_a/2}(y_a, g_x)$ are mutually disjoint;

(iii) $\sum_a r_a^{2n-3} \le 1$;

(iv) The number of overlapping balls $B_{2r_a}(y_a, g_x)$ is uniformly bounded.

We denote by $\bar{\eta}$ a cut-off function: $\mathbb{R} \mapsto \mathbb{R}$ satisfying: $0 \le \bar{\eta} \le 1$, $|\bar{\eta}'(t)| \le 2$ and

$$\bar{\eta}(t) = 1 \ \text{ for } \ t > \frac{3}{2} \ \text{ and } \ \bar{\eta}(t) = 0 \ \text{ for } \ t \le 1.$$

Put

$$\chi_a(y) = \bar{\eta}\left(\frac{d(y, y_a)}{r_a}\right) \ \text{ if } y \in B_{2r_a}(y_a, g_x) \quad \text{and} \quad \chi_a(y) = 1 \ \text{ otherwise.}$$

Clearly, $\chi_a \equiv 0$ on $B_{r_a}(y_a, g_x)$. By (iv), near any y, the number of non-vanishing χ_a is uniformly bounded by A, so the product function $\chi = \prod_a \chi_a$ is smooth and vanishes near $\mathcal{S}_{x,2} \cap B_{\bar{\epsilon}^{-1}}(x, g_x)$, furthermore, we have

$$\int_{\mathcal{C}_x} |\nabla \chi|^2 \, \omega_x^n \, \leq \, A \sum_a \int_{B_{r_a}(y_a, g_x)} |\nabla \chi_a|^2 \, \omega_x^n \, \leq \, C \, \epsilon_0, \tag{5.9}$$

where C is a constant which depends on c and A.

We still denote by η the cut-off function given above. Now we put $\rho(y) = d(y, \mathcal{S}_{x,1})$. Now we define as follows: If $\rho(y) \geq \bar{\epsilon}/3$, put $\gamma_{\bar{\epsilon}}(y) = \chi(y)$ and if $\rho(y) < \bar{\epsilon}$, put

$$\gamma_{\bar{\epsilon}}(y) \, = \, \chi(y) \, \eta \left(\log \left(- \log \left(\frac{\rho(y)}{\bar{\epsilon}} \right) \right) \right). \tag{5.10}$$

Clearly, $\gamma_{\bar{\epsilon}}$ is smooth. If we choose ϵ_0 and δ sufficiently small, we have $\gamma_{\bar{\epsilon}}(y) = 1$ for any y with $d(y, \mathcal{S}_x) \geq \bar{\epsilon}$, also $\gamma_{\bar{\epsilon}}$ vanishes in a neighborhood of $\mathcal{S}_x$. Furthermore, by using (5.9), the Fubini theorem and our assumption $\mathbf{A}_1$, we can show

$$\int_{B_{\bar{\epsilon}^{-1}}(o, g_x)} |\nabla \gamma_{\bar{\epsilon}}|^2 \, \omega_x^n \, \leq \, C' \left(\epsilon_0 \, + \, \frac{1}{- \log \delta} \right),$$

where C' is a constant which may depend on $\bar{\epsilon}$. Then the lemma follows under Assumption $\mathbf{A}_1$ if ϵ_0 and δ are sufficiently small.

□

Now assuming Lemma 5.8, we prove the partial C^0-estimate.

First we define η to be a cut-off function satisfying:

$$\eta(t) \, = \, 1 \;\text{ for }\; t \, \leq \, 1, \quad \eta(t) \, = \, 0 \;\text{ for }\; t \, \geq \, 2 \;\text{ and }\; |\eta'(t)| \, \leq \, 1.$$

Choose $\bar{\epsilon}$ such that $V(x; \epsilon)$ contains the support of $\gamma_{\bar{\epsilon}}$ constructed in Lemma 5.8 and $\gamma_{\bar{\epsilon}} = 1$ on $V(x; \delta_0)$, where $\delta_0 > 0$ is determined later. Clearly, we can choose $\bar{\epsilon}$ as small as we want if ϵ is sufficiently small.

We define for any $y \in V(x; \epsilon)$

$$\tilde{\tau}(\phi(y)) \, = \, \eta(2\delta \rho_x(y)) \, \eta(2\delta \rho_x(y)^{-1}) \, \gamma_{\bar{\epsilon}}(y) \, \tau(\phi(y)).$$

Clearly, $\tilde{\tau}$ vanishes outside $\phi(V(x; \epsilon))$, therefore, it extends to a smooth section of $K_{M_\infty}^{-\ell}$ on M_∞. Furthermore, $\tilde{\tau}$ satisfies:

(i) $\tilde{\tau} = \tau$ on $\phi(V(x; \delta_0))$;

(ii) There is an $\nu = \nu(\delta, \epsilon)$ such that

$$\int_{M_\infty} ||\bar{\partial} \tilde{\tau}||^2 \, \omega_\infty^n \, \leq \, \nu \, r^{2n-2}.$$

Note that we can make ν as small as we want so long as δ, ϵ and $\bar{\epsilon}$ are sufficiently small.

Since $(M\backslash D, \omega_i)$ converge to $(M_\infty\backslash S, \omega_\infty)$ and the Hermitian metrics H_i on K_M^{-1} converge to H_∞ on $M_\infty\backslash S$ in the C^∞-topology. Therefore, there are diffeomorphisms

$$\tilde{\phi}_i : M_\infty\backslash\mathcal{S} \mapsto M\backslash T_i(D)$$

and smooth isomorphisms

$$F_i : K_{M_\infty}^{-\ell} \mapsto K_M^{-\ell}$$

over $M\backslash T_i(D)$, where $T_i(D)$ is the set of all points within distance δ_i from D with respect to the metric ω_i, where $\delta_i > 0$ and $\lim \delta_i = 0$, satisfying:

$\mathbf{C}_1$: $\tilde{\phi}_i(M_\infty\backslash T_{\delta_i}(\mathcal{S})) \subset M\backslash T_\delta(D)$, where $T_{\delta_i}(\mathcal{S}) = \{x \in M_\infty \mid d_\infty(x,\mathcal{S}) \le \delta_i\}$;

$\mathbf{C}_2$: $\pi_i \circ F_i = \tilde{\phi}_i \circ \pi_\infty$, where π_i and π_∞ are corresponding projections;

$\mathbf{C}_3$: $||\tilde{\phi}_i^* \omega_i - \omega_\infty||_{C^2(M_\infty\backslash T_{\delta_i}(\mathcal{S}))} \le \delta_i$;

$\mathbf{C}_4$: $||F_i^* H_i - H_\infty||_{C^4(M_\infty\backslash T_i(\mathcal{S}))} \le \delta_i$.

We may assume that i is sufficiently large so that $\phi(V(x;\epsilon)) \subset M\backslash T_i(\mathcal{S})$. Put $\tilde{\tau}_i = F_i(\tilde{\tau})$, then we deduce from the above

(i') $\tilde{\tau}_i = F_i(\tau)$ on $\tilde{\phi}_i(\phi(V(x;\delta_0)))$;

(ii') For i sufficiently large, we have

$$\int_{M_i} ||\bar{\partial}\tilde{\tau}_i||_i^2 \, \omega_i^n \le 2\nu \, r^{2n-2},$$

where $||\cdot||_i$ denotes the Hermitian norm corresponding to H_i.

By the L^2-estimate in Lemma 5.4, we get a section v_i of $K_{M_i}^{-\ell}$ such that

$$\bar{\partial} v_i = \bar{\partial}\tilde{\tau}_i$$

and

$$\int_{M_\infty} ||v_i||_i^2 \, \omega_i^n \le \frac{1}{\ell} \int_{M_i} ||\bar{\partial}\tilde{\tau}_i||_i^2 \, \omega_\infty^n \le 3\nu \, r^{2n}.$$

Here we have used the fact that ℓ is the integral part of r^{-2}.

Put $\sigma_i = \tilde{\tau}_i - v_i$, it is a holomorphic section of $K_{M_i}^{-\ell}$. By (i) and Lemma 5.7, the C^4-norm of $\bar{\partial} v_i$ on $\tilde{\phi}_i(\phi(V(x;\delta_0)))$ is bounded from above by $c\delta$ for a uniform constant c. By the standard elliptic estimates, we have

$$\sup_{\tilde{\phi}(\phi(V(x;2\delta_0)\cap B_1(o,g_x)))} ||v_i||_i^2 \le C\,(\delta_0 r)^{-2n} \int_{M_i} ||v_i||_i^2 \, \omega_i^n \le C\,\delta_0^{-2n}\,\nu.$$

Here C denotes a uniform constant. For any given δ_0, if δ and ϵ are sufficiently small, then we can make ν such that

$$8C\,\nu \le \delta_0^{2n}.$$

Then we can deduce from the above estimates

$$||\sigma_i||_i \,\geq\, ||F_i(\tau)||_i \,-\, ||v_i||_i \,\geq\, \frac{1}{2} \quad \text{on} \;\; \tilde{\phi}_i(\phi(V(x;\delta_0) \cap B_1(o, g_x))).$$

On the other hand, by applying the derivative estimate in Corollary 4.2 to σ_i, we get

$$\sup_{M_i} ||\nabla \sigma_i||_i \,\leq\, C' \ell^{\frac{n+1}{2}} \left(\int_{M_i} ||\sigma_i||_i^2 \, \omega_i^n \right)^{\frac{1}{2}} \,\leq\, C' \, r^{-1}.$$

Since the distance $d(x, \phi(\delta_0 u))$ is less than $10\delta_0 r$ for some $u \in \partial B_1(o, g_x)$, if i is sufficiently large, we deduce from the above estimates

$$||\sigma_i||_i(x_i) \,\geq\, 1/4 - C' \, \delta_0,$$

hence, if we choose δ_0 such that $C'\delta_0 < 1/8$, then $\rho_{\omega_i,\ell}(x_i) > 1/8$. Theorem 1.2, i.e., the partial C^0-estimate for conic Kähler-Einstein metrics, is proved.

As indicated in [Ti09] for smooth Kähler-Einstein metrics, by the arguments in the proof of the partial C^0-estimate, we can prove the following regularity for M_∞:

Theorem 5.9. *The Gromov-Hausdorff limit M_∞ is a normal variety embedded in some $\mathbb{C}P^N$ and S is a subvariety consisting a divisor D_∞ and a subvariety of complex codimension at least 2. Moreover, D_∞ is the limit of D under the Gromov-Hausdorff convergence.*

Proof. For the readers' convenience, we include a proof. Let us recall some well-known facts (cf, [Ti09]). For any i and sufficiently large ℓ, we can choose an orthonormal basis $\{\sigma_{i,\ell}\}$ of $H^0(M, K_M^{-\ell})$ with respect to ω_i and use this to define a Kodaira embedding

$$\psi_{i,\ell} : M \,\mapsto\, \mathbb{C}P^{N_\ell}, \quad \text{where } N_\ell + 1 \,=\, \dim H^0(M, K_M^{-\ell}).$$

By using the L^2-estimate for $\bar\partial$-operator, we can find an exhaustion of $M_\infty \backslash S$ by open subsets $V_1 \subset V_2 \subset \cdots \subset V_\ell \subset \cdots$ such that $\psi_{i,\ell}$ converge to an embedding

$$\psi_{\infty,\ell} : V_\ell \subset M_\infty \,\mapsto\, \mathbb{C}P^{N_\ell}.$$

By the partial C^0-estimate, there is an integer $m > 0$ such that for any $\ell = mk$, $\psi_{i,\ell}$ converge to an extension of $\psi_{\infty,\ell}$ on M_∞ under the convergence of (M, ω_i) to $(M_\infty, \omega_\infty)$. We still denote this extension by

$$\psi_{\infty,\ell} : \, M_\infty \,\mapsto\, \mathbb{C}P^{N_{\ell_a}}.$$

By the estimate in Corollary 4.2, $\psi_{i,\ell}$ are uniformly Lipschtz, so $\psi_{\infty,\ell}$ is a Lipschtz map.

Claim: M_∞ is a variety.

For this, we only need to show that for $k \geq n+1$, $\psi_{\infty,\ell}$ is a homeomorphism from M_∞ onto its image which is also the limit of complex submanifolds $\psi_{i,\ell}(M) \subset \mathbb{C}P^{N_\ell}$.

By the same arguments as those in proving the partial C^0-estimate, for any $r > 0$, there are $k(r)$ and $s(k)$ such that if $k \geq k(r)$, then for any $x, y \in M$ such that $d_i(x,y) \geq r$, where $d_i(\cdot,\cdot)$ denotes the distance of the metric ω_i, there is a holomorphic section $\varsigma_i \in H^0(M, K_M^{-\ell})$, where $\ell = mk$, satisfying:

$$\int_M ||\varsigma_i||_i^2 \omega_i^n = 1 \quad \text{and} \quad |||\varsigma_i||_i(x) - ||\varsigma_i||_i(y)| \geq s(k). \tag{5.11}$$

The above claim follows from this and the effective finite generation of the anti-canonical ring of M as shown in the thesis of Chi Li [Li12]. [15] For the orthonormal basis $\{\sigma_{i,a}\}_{0\leq a\leq N_m}$ of $H^0(M, K_M^{-m})$ with respect to ω_i, by the partial C^0-estimate and Corollary 4.2, we have

$$c(m) \leq \sum_{a=0}^{N_m} ||\sigma_{i,a}||_i^2 \leq c(m)^{-1}, \tag{5.12}$$

where $c(m)$ is a uniform constant independent of i.

Lemma 5.10. *For any $l \geq 1$ and $\varsigma \in H^0(M, K_M^{-(n+1+l)m})$, there are $h_0, \cdots, h_{N_m}$ in $H^0(M, K_M^{-(n+l)m})$ satisfying:*

$$\varsigma = \sum_{a=0}^{N_m} h_a\, \sigma_{i,a} \quad \text{and} \quad \int_M ||h_a||_i^2\, \omega_i^n \leq C(m,l) \int_M ||\varsigma||_i^2\, \omega_i^n, \tag{5.13}$$

where $C(m,l)$ is a constant depending only on $c(m)$, l and n.

This can be proved by using the Skoda-Siu estimate, now a standard technique (cf. [Li12], Proposition 7).

Note that for any $x \in M_\infty$ and $k \geq 1$, we have

$$\psi_{\infty,mk}^{-1}(\psi_{\infty,mk}(x)) \subseteq \psi_{\infty,m}^{-1}(\psi_{\infty,m}(x)). \tag{5.14}$$

Using this and Lemma 5.10 with $i \to \infty$, we get

$$\psi_{\infty,m(n+1+l)}^{-1}(\psi_{\infty,m(n+1+l)}(x)) \supseteq \psi_{\infty,m(n+1)}^{-1}(\psi_{\infty,m(n+1)}(x)).$$

It follows from (5.11) that for any $x \neq y \in M_\infty$,

$$\psi_{\infty,m(n+1+l)}(x) \neq \psi_{\infty,m(n+1+l)}(y)$$

if l is sufficiently large. Therefore, we can get

$$\psi_{\infty,m(n+1)}(x) \neq \psi_{\infty,m(n+1)}(y).$$

This implies that $\psi_{\infty,m(n+1)}$ is a homeomorphism, so M_∞ is a variety.

[15] As I advocated in many occasions before (cf. [Ti09]), the partial C^0-estimate corresponds to an effective version of the finite generation of the anti-canonical ring. Chi Li showed precisely in [Li12] how this works.

There is another way of proving that $\psi_{\infty,mk}$ is a homeomorphism for k sufficiently large. By (5.14), the composition $\psi_{\infty,m} \cdot \psi^{-1}_{\infty,mk}$ is a well-defined map from the variety Y_{mk} onto Y_m, where

$$Y_{mk} \,=\, \lim_{i\to\infty} \psi_{i,mk}(M) \subset \mathbb{C}P^{N_{mk}}, \quad Y_m \,=\, \lim_{i\to\infty} \psi_{i,m}(M) \subset \mathbb{C}P^{N_m}.$$

Moreover, this map is also the limit of holomorphic maps $\psi_{i,m} \cdot \psi^{-1}_{i,mk}$, so it is a holomorphic map. Since $\psi_{\infty,m}$ restricted to V_m is an embedding for m sufficiently large, we know that $\psi_{\infty,mk}(\psi^{-1}_{\infty,m}(z))$ is either a point or a connected subvariety in the complex limit space Y_{mk}. The second case can be ruled out by using the fact that there is a bounded function u such that

$$\frac{1}{mk}\omega_{FS}|_{Y_{mk}} \,=\, \frac{1}{m}(\psi_{\infty,m} \cdot \psi^{-1}_{\infty,mk})^*(\omega_{FS}|_{Y_m}) \,+\, \sqrt{-1}\partial\bar\partial u,$$

where ω_{FS} always denotes the Fubini-Study metric.

Next we prove that M_∞ is normal. This means that $M_\infty \backslash S$ is locally connected. If $\beta_\infty = 1$, it is trivially true since the singular set of M_∞ is of complex codimension at least 2. So we may assume $\beta_\infty < 1$. There are several approaches. One can use a local version of the Cheeger-Gromoll splitting theorem (cf. [An90]). One can also generalize the arguments I had in [Ti89] or use the Cheeger-Colding theory.

Before we prove the normality of M_∞, we make a remark: By Corollary 4.2 and the partial C^0-estimate, log $\rho_{\omega_i,m}$ converge to a uniformly continuous function log $\rho'_{\infty,m}$ on M_∞. This implies that ω_∞ is the curvature of a continuous Hermitian metric on $K^{-1}_{M_\infty}$, so $||\cdot||_\infty$ is a continuous Hermitian metric on M_∞ even when $\beta_\infty < 1$. Therefore, the defining section σ_i of D normalized by ω_i converge to a holomorphic section σ_∞ of $K^{-\lambda}_{M_\infty}$. Clearly, the singular set $\mathcal{S}$ of $(M_\infty, \omega_\infty)$ is the divisor D_∞ defined by σ_∞ possibly plus a closed subset $\mathcal{S}_{2n-4}$ of complex codimension at least 2.

Therefore, if M_∞ is not normal, then $M_\infty\backslash D_\infty$ is not locally connected near a point, say x, in D_∞. Since $x \in \mathcal{S}\backslash\bar{\mathcal{S}}_{2n-4}$, there is a tangent cone $\mathcal{C}_x$ of M_∞ at x of the form $\mathbb{C}^{n-1}\times\mathcal{C}'_x$, where $\mathcal{C}'_x$ is a 2-dimensional flat cone of angle $2\pi\bar\beta$, where $(1-\bar\beta) = k(1-\beta_\infty)$. However, $\mathcal{C}_x\backslash\mathcal{S}_x$ is connected, so $M_\infty\backslash D_\infty$ is connected near x, a contradiction. Therefore, M_∞ must be normal.

Note that the normality also follows from a result of Colding-Naber who proved the convexity of $M_\infty\backslash\mathcal{S}$.

□

Of course, one can further analyze the finer asymptotic structure of ω_∞ along D_∞. By the partial C^0-estimate and Corollary 4.2, we have

$$\omega_\infty \,\ge\, c\,\psi^*_{\infty,\ell}(\omega_{FS}),$$

where $\ell = mk$ and c is some positive constant. Using this, when $\beta_\infty < 1$, one can show that ω_∞ is a conic Kähler-Einstein metric with conic angle $2\pi\bar\beta$ along

D_∞ in a weaker sense, where $(1-\bar\beta) = k(1-\beta_\infty)$[16]. It is an interesting problem to examine the precise behavior of ω_∞ along D_∞.

The following theorem may be useful in the future.

Theorem 5.11. *For each $\ell > 0$, let $\{\sigma_{i,\alpha}\}$ be an orthonormal basis of $H^0(M, K_M^{-\ell})$. Then by taking a subsequence if necessary, $\{\sigma_{i,\alpha}\}$ converge to an orthonormal basis $\{\sigma_{\infty,\alpha}\}$ of $H^0(M_\infty, K_{M_\infty}^{-\ell})$. In particular, it implies that $H^0(M_\infty, K_{M_\infty}^{-\ell})$ is of finite dimension and $\rho_{\omega_i,\ell}$ converge to $\rho_{\omega_\infty,\ell}$ as i tends to ∞.*

Proof. The arguments appeared before (cf. [Ti09]) and are based on the L^2-estimate for the $\bar\partial$-operator. In view of Lemma 5.3, it suffices to prove that any τ in $H^0(M_\infty, K_{M_\infty}^{-\ell})$ with its L^2-norm being one is a limit of a sequence $\tau_i \in H^0(M, K_M^{-\ell})$. We will adopt the notations in establishing of the partial C^0-estimate, particularly, $\mathbf{C}_1$-$\mathbf{C}_4$.

The following lemma is an analogue of Lemma 5.8. It is easy to prove by using Theorem 5.9.

Lemma 5.12. *For any $\epsilon > 0$, there is a smooth function γ_ϵ on M_∞ satisfying:*

(1) $\gamma_\epsilon(x) = 1$ for any x with $d_\infty(x, \mathcal{S}) \ge \epsilon$;

(2) $0 \le \gamma_\epsilon \le 1$ and $\gamma_\epsilon(x) = 0$ in an neighborhood of $\mathcal{S}$;

(3) $|\nabla\gamma_\epsilon| \le C$ for some constant $C = C(\epsilon)$ and

$$\int_{M_\infty} |\nabla\gamma_\epsilon|^2\,\omega_\infty^n \le \epsilon.$$

For each i and $\epsilon \in (0,1)$, define

$$\xi_\epsilon(x) = F_i^*(\gamma_\epsilon\,\tau)(x),$$

Then ξ_ϵ is a smooth section of $K_M^{-\ell}$ satisfying:

(1) $\xi_\epsilon(x) = 0$ in an neighborhood of S;

(2) put $\zeta_\epsilon = \bar\partial\xi_\epsilon$, then

$$\int_M ||\zeta_\epsilon||^2\,\omega_i^n \le 2\int_M ||\bar\partial(F_i^*\tau)||^2\,\omega_i^n + C\,\epsilon\,\sup_{M_\infty}||\tau||_\infty^2, \tag{5.15}$$

where C is a uniform constant.

Let δ_i be given in $\mathbf{C}_1$-$\mathbf{C}_4$. Then there are ϵ_i with $\lim\epsilon_i = 0$ such that γ_{ϵ_i} in the above lemma vanishes in an neighborhood of the closure of $T_{\delta_i}(\mathcal{S})$. Put $\xi_i = \xi_{\epsilon_i}$ and $\zeta_i = \zeta_{\epsilon_i}$, then it follows from (5.15) that

$$\lim_{i\to\infty}\int_M ||\zeta_i||^2\,\omega_i^n = 0. \tag{5.16}$$

[16] The integer k may vary on different connected components of D_∞.

Applying Lemma 5.4 to ζ_i, we get ϑ_i such that $\bar\partial \vartheta_i = \zeta_i$ and

$$\int_M ||\vartheta_i||_i^2\, \omega_i^n \,\le\, \frac{1}{\ell+\mu}\int_M ||\zeta_i||_i^2\, \omega_i^n \,\to\, 0.$$

On the other hand, $\zeta_i = \bar\partial \xi_i$. By the construction of ξ_i, we can easily show that ξ_i converge to τ in the C^∞-topology outside $\mathcal{S}$ and

$$\lim_{i\to\infty}\int_M ||\xi_i||_i^2\, \omega_i^n \,=\, \int_{M_\infty} ||\tau||_\infty^2\, \omega_\infty^n \,>\, 0.$$

Then $\tau_i = \xi_i - \vartheta_i$ defines a holomorphic section of $K_M^{-\ell}$ which converges to τ in the L^2-topology. By the standard elliptic estimates, we can easily show that τ_i converge to τ in the C^∞-topology outside $\mathcal{S}$. This proves Theorem 5.11. $\square$

6 Proving Theorem 1.1

In this section, we complete the proof of Theorem 1.1, i.e., if a Fano manifolds M is K-stable, then it admits a Kähler-Einstein metric. In fact, as I pointed out in describing my program on the existence of Kähler-Einstein metrics, the reduction of Theorem 1.1 from the partial C^0-estimate had been known to me for long. [17]

As explained in the introduction, in order to prove Theorem 1.1, we only need to establish the C^0-estimate for the solutions of the complex Monge-Ampere equations for $\beta > 1-\lambda^{-1}$:

$$(\omega_\beta + \sqrt{-1}\partial\bar\partial\varphi)^n \,=\, e^{h_\beta - \mu\varphi}\omega_\beta^n, \tag{6.1}$$

where ω_β is a suitable family of conic Kähler metrics with $[\omega_\beta] = 2\pi c_1(M)$ and cone angle $2\pi\beta$ along D and h_β is determined by

$$\mathrm{Ric}(\omega_\beta) \,=\, \mu\omega + 2\pi(1-\beta)[D] + \sqrt{-1}\partial\bar\partial h_\beta \quad\text{and}\quad \int_M (e^{h_\beta}-1)\omega_\beta^n == 0.$$

By the discussed in the introduction, we know that there is a non-empty and maximal interval $E = (1-\lambda^{-1}, \bar\beta)$ for some $\bar\beta \in (1-\lambda^{-1}, 1)$ or $(1-\lambda^{-1}, 1]$ such that (6.1) has a solution φ_β for any $\beta \in E$. Actually, such a solution φ_β is unique, so $\{\varphi_\beta\}$ is a continuous family on M and smooth outside D.[18] If $1 \in E$, we already have Theorem 1.1 and nothing more needs to be done. Hence, we may assume that $E = (1-\lambda^{-1}, \bar\beta)$ for some $\bar\beta < 1$, we will derive a contradiction. By our assumption and the results in [JMR11], $||\varphi_\beta||_{C^0}$ diverge to ∞ as β tends to $\bar\beta$. We will show that it contradicts to the K-stability of M. Now let us recall the definition of the K-stability. I will use the original one from

[17] Our program was originally proposed for the Aubin continuity method, but it works for the new Donaldson-Li-Sun continuity method in an identical way.

[18] In fact, one can use prove this continuity and smoothness directly by using the Inverse Function Theorem as we argued for the openness of E.

[Ti97] which is directly related to our program of establishing the existence of Kähler-Einstein metrics through the continuity method.

First we recall the definition of the Futaki invariant [Fu83]: Let M_0 be any Fano manifold and ω be a Kähler metric with $c_1(M)$ as its Kähler class, for any holomorphic vector field X on M_0, Futaki defined

$$f_{M_0}(X) \,=\, \int_M X(h_\omega)\,\omega^n, \tag{6.2}$$

where $\mathrm{Ric}(\omega) - \omega = \sqrt{-1}\partial\bar\partial h_\omega$. Futaki proved in [Fu83] that $f_M(X)$ is independent of the choice of ω, so it is a holomorphic invariant. In [DT92], the Futaki invariant was extended to normal Fano varieties. The extension is based on the following reformulation:

$$f_{M_0}(X) \,=\, -\,n\int_M \theta_X\,(\mathrm{Ric}(\omega)-\omega)\wedge\omega^{n-1}, \tag{6.3}$$

where $i_X\omega = \sqrt{-1}\,\bar\partial\theta_X$.

Now let M be a Fano manifold M. By the Kodaira embedding theorem, for ℓ sufficiently large, any basis of $H^0(M, K_M^{-\ell})$ gives an embedding

$$\phi_\ell : M \mapsto \mathbb{C}P^N,$$

where $N = \dim_{\mathbb{C}} H^0(M, K_M^{-\ell}) - 1$. Any other basis gives an embedding of the form $\sigma\circ\phi_\ell$, where $\sigma \in G = \mathbf{SL}(N+1,\mathbb{C})$.

For any algebraic subgroup $G_0 = \{\sigma(t)\}_{t\in\mathbb{C}^*}$ of $\mathbf{SL}(N+1,\mathbb{C})$, there is a unique limiting cycle

$$M_0 \,=\, \lim_{t\to 0}\sigma(t)(M) \subset \mathbb{C}P^N.$$

Let X be the holomorphic vector field whose real part generates the action by $\sigma(e^{-s})$. By [DT92], if M_0 is normal, there is a generalized Futaki invariant $f_{M_0}(X)$ defined by (6.3).

Now we can introduce the K-stability from [Ti97].

Definition 6.1. *We say that M is K-stable with respect to $K_M^{-\ell}$ if $f_{M_0}(X)\ge 0$ for any $G_0\subset \mathbf{SL}(N+1)$ with a normal M_0 and the equality holds if and only if M_0 is biholomorphic to M. We say that M is K-stable if it is K-stable for all sufficiently large ℓ.*

There are other formulations of the K-stability by S. Donaldson in [Do02] and S. Paul in [Pa08].

It was proved in [Ti97]

Theorem 6.2. *Let M be a Fano manifold without non-trivial holomorphic vector fields and which admits a Kähler-Einstein metric. Then M is K-stable.*

Now we return to our Fano manifold M in Theorem 1.1 and those solutions φ_β ($\beta\in E$) as above. In order to get a contradiction, we need to produce only

a normal Fano variety M_0 as in Definition 6.1 and with non-positive Futaki invariant.

Let $\{\beta_i\}$ be a sequence with $\lim \beta_i = \bar{\beta}$. Write $\varphi_i = \varphi_{\beta_i}$. If $\sup_M \varphi_i$ is uniformly bounded, by the Harnack-type estimate in Theorem in [JMR11], the C^0-norm of φ_i is uniformly bounded. So, by [JMR11] again, φ_i converge to a solution of (6.1) for $\beta = \bar{\beta}$. A contradiction! Therefore, we have

$$\lim_{i\to\infty} \sup_M \varphi_i = \infty.$$

We will fix such a sequence $\{\beta_i\}$ and write

$$\omega_i = \omega + \sqrt{-1}\,\partial\bar{\partial}\varphi_i.$$

Then ω_i is a conic Kähler-Einstein metric on M with cone angle $2\pi\beta_i$ along D. By taking a subsequence of necessary, we may assume that (M, D, ω_i) converge to $(M_\infty, D_\infty, \omega_\infty)$ in the Gromov-Hausdorff topology. By Theorem 5.9, M_∞ is a normal subvariety in some projective space $\mathbb{C}P^N$ and ω_∞ is a smooth Kähler-Einstein metric outside a divisor D_∞ and the singular set $\bar{\mathcal{S}}$ of M_∞.[19] We will identify M_∞ with its image in $\mathbb{C}P^N$ by an embedding defined by a basis $\{S_{\infty,\alpha}\}$ of $H^0(M_\infty, K_{M_\infty}^{-\ell})$, in fact, such a basis $\{S_{\infty,\alpha}\}$ is orthonormal with respect to the inner product on $H^0(M_\infty, K_{M_\infty}^{-\ell})$ by ω_∞.

Similarly, we embed M by orthonormal bases of $H^0(M, K_M^{-\ell})$ with respect to ω_i. All these embeddings differ by transformations in G. On the other hand, by taking a subsequence if necessary, we may assume that those orthonormal bases of $H^0(M, K_M^{-\ell})$ converge to the orthonormal basis $\{S_{\infty,\alpha}\}$ of $H^0(M_\infty, K_{M_\infty}^{-\ell})$ under the convergence of (M, D, ω_i) to $(M_\infty, D_\infty, \omega_\infty)$. It implies that (M_∞, D_∞) lies in the closure of the orbit of (M, D) under the group action of $G = \mathbf{SL}(N+1, \mathbb{C})$ on $\mathbb{C}P^N$. Then one can deduce from some general facts in algebraic geometry that the stabilizer G_∞ of M_∞ in G contains a holomorphic subgroup.[20] We need to prove that it contains a $\mathbb{C}^*$-subgroup. Then, using the Kähler-Einstein metric ω_∞, one can show that the generalized Futaki invariant is not positive. This contradicts to the K-stability.

Lemma 6.3. *The Lie algebra η_∞ of G_∞ is reductive.*

Proof. The arguments are standard. Let $X \in \eta_\infty$, i.e., a holomorphic vector field on $\mathbb{C}P^N$ which is tangent to M_∞, then there is a smooth function θ such that $i_X\omega_{FS} = \ell\sqrt{-1}\,\bar{\partial}\theta$. We have

$$\ell\,\omega_\infty = \omega_{FS}|_{M_\infty} + \sqrt{-1}\,\partial\bar{\partial}\rho_{\omega_\infty,\ell}.$$

It follows

$$i_X\omega_\infty = \sqrt{-1}\,\bar{\partial}\theta_\infty, \quad \text{where } \theta_\infty = \theta + \frac{1}{\ell}X(\rho_{\omega_\infty,\ell}).$$

[19] We have seen in last section that ω_∞ has locally continuous potentials.

[20] For the Aubin continuity, one can show by geometric analytic arguments that M_∞ admits a $\mathbb{C}^*$-action. One should be able to extend this method to the continuity method proposed by Donaldson et al.

It is a fact that X generates a $\mathbb{C}^*$-action if and only if it is a complexication of a Killing field. Therefore, if we normalize X by multiplication by a complex number such that $\sup_{M_\infty} \theta_\infty = 1$, we want to show that the imaginary part of X is Killing. The standard computations show that if θ_∞ is normalized by

$$\int_{M_\infty} \theta_\infty \, \omega_\infty^n \, = \, 0,$$

then

$$\Delta_\infty \theta_\infty \, + \, \mu_\infty \, \theta_\infty \, = \, 0 \quad \text{on } M_\infty \backslash D_\infty \cup \bar{\mathcal{S}},$$

where Δ_∞ denotes the Laplacian of ω_∞ and $\mu_\infty = 1 - (1 - \bar{\beta})\lambda$. On the other hand, by using our estimates on $\rho_{\omega_\infty, \ell}$ and the Bochner identity, we can show that θ_∞ is Lipschtz continuous, thus it extends to an eigenfunction of Δ_∞, so do its real and imaginary parts. It follows from the standard arguments that the imaginary part of θ_∞ induces a Killing field. Then the lemma is proved. $\square$

As observed in [Do11] and [Li11]), by using the same arguments as in [Fu83], one can define the Futaki invariant $f_{M_\infty, (1-\beta) D_\infty}(X)$, also referred as the log-Futaki invariant, for conic Kähler metrics on M_∞ with cone angle $2\pi\beta$ along D_∞ ($\beta \in (0, 1)$). Furthermore, if there is a conic Kähler-Einstein metric with angle $2\pi\beta$ along D_∞, the log-Futaki $f_{M_\infty, (1-\beta) D_\infty}$ vanishes. In our case, though ω_∞ may not be smooth along D_∞ even in the conic sense, using the Lipschtz continuity of θ_∞, one can still prove the vanishing of $f_{M_\infty, (1-\beta) D_\infty}(X)$ by the same arguments as in the smooth case. Then the Futaki invariant $f_{M_\infty}(X) \, \leq \, 0$. This can be derived by using the formula (cf. [Li11], [Su11]):[21]

$$0 \, = \, f_{M_\infty, (1-\beta) D_\infty}(X) \, = \, f_{M_\infty}(X) \, + \, (1 - \beta) \int_{D_\infty} \theta_\infty \, d\mathcal{H}^{2n-2},$$

where $d\mathcal{H}^{2n-2}$ denotes the (2n-2)-dimensional Hausdorff measure on D_∞ induced by ω_∞. To see this, we first observe that $f_{M_\infty, (1-\beta_1) D_\infty}(X) > 0$ for some $\beta_1 \in (1 - \lambda^{-1}, \beta)$, e.g., if it is sufficiently close to $1 - \lambda^{-1}$ because there is a corresponding conic Kähler-Einstein metric with angle $2\pi\beta_1$, on the other hand, because of the linearity, we have

$$(\beta - \beta_1) \, f_{M_\infty}(X) \, = \, (1 - \beta_1) \, f_{M_\infty, (1-\beta) D_\infty}(X) \, - \, (1 - \beta) \, f_{M_\infty, (1-\beta_1) D_\infty}(X),$$

hence, $f_{M_\infty}(X) \leq 0$.

On the other hand, by our assumption that M is K-stable, since M_∞ is not biholomorphic to M,

$$f_{M_\infty}(X) > 0.$$

This is a contradiction! Therefore, φ_β are uniformly bounded and consequently, $\bar{\beta} \in E$, so E is closed and Theorem 1.1 is proved.

There is another way of finishing the proof of Theorem 1.1 by using the CM-stability introduced in [Ti97]. The CM-stability can be regarded as a geometric

[21] Chi Li pointing out that this formula first appeared in [Do11]. I thank him for this as well as some other inputs on log-Futaki invariants.

invariant theoretic version of the K-stability. It follows from [PT06] and [Pa08] that the CM-stability is equivalent to the K-stability. In the following, we outline this alternative proof of Theorem 1.1.

Let us recall the CM-stability. We fix an embedding $M \subset \mathbb{C}P^N$ by $K_M^{-\ell}$ as above. Let $\pi : \mathcal{X} \mapsto Z$ be the universal family of n-dimensional normal varieties [22] in $\mathbb{C}P^N$ with the same Hilbert polynomial as that of M. Clearly, $G = \mathbf{SL}(N+1)$ acts both $\mathcal{X}$ and Z such that π is equivariant.

Consider the virtual bundle

$$\mathcal{E} \,=\, (n+1)(\mathcal{K} - \mathcal{K}^{-1})(\mathcal{L} - \mathcal{L}^{-1})^n - n(\mathcal{L} - \mathcal{L}^{-1})^{n+1},$$

where $\mathcal{K} \,=\, K_{\mathcal{X}} \otimes K_Z^{-1}$ is the relative canonical bundle and $\mathcal{L}$ is the pull-buck of the hyperplane line bundle on $\mathbb{C}P^N$.

Let L be the determinant line bundle $\det(\mathcal{E}, \pi)$. Clearly, G acts naturally on the total space of L.

Definition 6.4. *Let $z = \pi(M)$ and $\tilde{z}$ be a non-zero lifting of z in the total space of L. We call M CM-stable with respect to $K_M^{-\ell}$ if the orbit $G \cdot \tilde{z}$ in the total space of L is closed and the stabilizer G_z of z is finite. We call M CM-semistable if 0 is not in the closure of $G \cdot \tilde{z}$. We call M CM-stable if it does with respect to all sufficiently large ℓ.*

Now we fix M, M_∞, ℓ as above. Given any $\sigma \in G$, there is an induced Kähler potential φ_σ

$$\frac{1}{\ell}\,\sigma^* \omega_{FS} \,=\, \omega_0 \,+\, \sqrt{-1}\,\partial\bar{\partial}\varphi_\sigma.$$

Define a functional on the orbit $G \cdot z$:

$$F_\ell(\sigma) \,=\, \mathbf{F}_{\omega_0}(\varphi_\sigma).$$

Then we have the following ([Ti97], Theorem 8.10)

Theorem 6.5. *The functional F_ℓ is proper on $G \cdot z \subset Z$ if and only if M is CM-stable with respect to $K_M^{-\ell}$.*

By our discussions in Section 3, we can show that $\mathbf{F}_{\omega_0,\mu}$ restricted to $G \cdot z$ is proper for any $\mu \in (0,1]$. Combining this properness with the partial C^0-estimate, we can bound the C^0-norm of φ_β in a uniform way. Then it follows from [JMR11] that E is closed. Therefore, we have proved

Theorem 6.6. *Let M be a Fano manifold without non-trivial holomorphc fields, then M admits a Kähler-Einstein metric if and only if M is CM-stable.*

In view of [PT06] and [Pa08], particularly Theorem D in [Pa08], this implies Theorem 1.1.

[22] Normality is not needed, but we assume this for simplicity. Also by [LX11], this assumption does not put any constraints on our results.

7 Appendix: The proof of Lemma 5.8

In this appendix, we complete the proof of Lemma 5.8. We will adopt the notations in Section 5, particularly, in the proof of those special cases of Lemma 5.8. The arguments of our proof are based on known techniques, though tedious. Note that if $\beta_\infty = 1$, then there is nothing to be proved since the singular set $\mathcal{S}_x$ is of complex dimension at least 2. So we may assume that $\beta_\infty < 1$. In this case, $\mathcal{S}_x$ has a decomposition into $\mathcal{S}_x^0$ and $\bar{\mathcal{S}}_x$ as before, and for any $y \in \mathcal{S}_x^0$, there is a tangent cone of $\mathcal{C}_x$ at y of the form $\mathbb{C}^{n-1} \times \mathcal{C}'_y$ for which Lemma 5.8 has been proved.

Fix any $y \in \mathcal{S}_x^0 \subset \mathcal{C}_x$, we have a tangent cone of the form $\mathbb{C}^{n-1} \times \mathcal{C}'_y$ at y, where $\mathcal{C}'_y$ denotes the standard 2-dimensional cone with angle $2\pi\bar{\beta}$, where $\bar{\beta} = \mu_a$ is given as in Lemma 5.5 and satisfies $(1-\bar{\beta}) = k(1-\beta_\infty)$ for some integer k.

There are $x_i \in M$ and $r_i > 0$ such that $(M, r_i^{-2}\omega_i, x_i)$ converge to the cone $(\mathcal{C}_x, g_x, o)$ in the Gromov-Hausdorff topology and smooth topology outside the singular set $\mathcal{S}_x$, in particular, there are diffeomorphisms

$$\tilde{\phi}_i : V(x;\delta_i) \mapsto M \backslash T_{\delta_i}(D),$$

where $T_{\delta_i}(D)$ is the set of all points within distance δ_i from D with respect to the metric ω_i and $\lim \delta_i = 0$, satisfying:

$$||r_i^{-2}\, \tilde{\phi}_i^* \omega_i - \omega_x||_{C^2(V(x;\delta_i))} \le \delta_i.$$

Furthermore, we may assume

$$B_{\frac{r_i}{2\delta_i}}(x_i, \omega_i) \backslash T_{\delta_i}(D) \subset \phi_i(V(x;\delta_i)).$$

Without loss of generality, we may assume that $\ell_i = r_i^{-2}$ are integers and Lemma 5.7 holds for such ℓ_i's.

Note that there is a tangent cone of the form $\mathbb{C}^{n-1} \times \mathcal{C}'_y$ with the standard cone metric $g_{\bar{\beta}}$ in the proof of Lemma 5.8. The singular set of this tangent cone is $\mathbb{C}^{n-1} \times \{0\}$. Therefore, there are integers $k_j = s_j^{-2}$ such that $(\mathcal{C}_x, k_j g_x, y)$ converge to $(\mathbb{C}^{n-1} \times \mathcal{C}'_y, g_{\bar{\beta}}, o)$ in the Gromov-Hausdorff topology and smooth topology outside the singular set. This implies that there are diffeomorphisms

$$\vartheta_j : U_j \mapsto \mathcal{C}_x \backslash \mathcal{S}_x$$

satisfying:

$$||s_j^{-2} \vartheta_j^* \omega_x - \omega_{\bar{\beta}}||_{C^2(U_j)} \le \frac{1}{j},$$

where

$$U_j = \{(z', z_n) \in \mathbb{C}^{n-1} \times \mathcal{C}'_y \mid |z'| < 9, \quad \frac{1}{j} < |z_n|^{\bar{\beta}} < 9\}.$$

We may also have

$$B_{9s_j}(y, g_x) \backslash T_{\frac{2}{j}}(\mathcal{S}_x) \subset \vartheta_j(U_j).$$

Combining these, we see that for any $\epsilon > 0$, there are j_ϵ and i_ϵ such that for any $j \geq j_\epsilon$ and $i \geq i_\epsilon$, the compositions

$$\tilde{\phi}_i \cdot \vartheta_j : U_j \mapsto M \backslash D$$

satisfying:

$$B_{(9-\epsilon)s_j r_i}(x_i, \omega_i) \backslash T_{\delta_i}(D) \subset \tilde{\phi}_i(\vartheta_j(U_j)) \subset B_{13 s_j r_i}(x_i, \omega_i)$$

and

$$||k_j \ell_i \, \vartheta_j^* \tilde{\phi}_i^* \omega_i - \omega_{\bar{\beta}}||_{C^2(U_j)} \leq \epsilon$$

Furthermore, by using the above arguments in establishing the partial C^0-estimate, given any finitely many holomorphic functions f_b ($b = 0, 1, \cdots, m$) with

$$\int_{\mathbb{C}^{n-1} \times \mathcal{C}'_y} |f_b|^2 \, e^{-\frac{|z'|^2 + |z_n|^{2\bar{\beta}}}{2}} \, \omega_{\bar{\beta}}^n = 1,$$

where $\omega_{\bar{\beta}}$ is the Kähler form of $g_{\bar{\beta}}$, we can construct holomorphic sections $S_{i,j}^b$ of $K_M^{-k_j \ell_i}$ over M such that

$$\sup_{U_j} |(\psi_{i,j})^*(S_{i,j}^b) - f_b| \leq \frac{\epsilon}{2},$$

where $\psi_{i,j}$ is the isomorphism constructed by Lemma 5.5 over U_j. By Corollary 4.2, for some uniform constant C, we have

$$||\nabla S_{i,j}^b||_i \leq C.$$

Now we take f_0 to be a positive constant function, then $S_{i,j}^0$ is almost a positive constant on $\tilde{\phi}_i(\vartheta_j(U_j))$ which contains $B_{8 s_j r_i}(x_i, \omega_i)$. Then by rechoosing j_ϵ and i_ϵ if necessary, we can deduce from the properties of $S_{i,j}^b$:

1. $B_{8 s_j r_i}(x_i, \omega_i)$ is contained in some $\mathbb{C}^{N'}$, where N' may depend on i, j;

2. There is a holomorphic map $F_{i,j}^m : \tilde{\phi}_i(\vartheta_j(U_j)) \mapsto \mathbb{C}^m$, where

$$F_{i,j}^m = \left(\frac{S_{i,j}^1(x)}{S_{i,j}^0(x)}, \cdots, \frac{S_{i,j}^m(x)}{S_{i,j}^0(x)} \right)$$

satisfying:

$$\left| F_{i,j}^m(\tilde{\phi}_i(\vartheta_j(z))) - \left(\frac{f_1}{f_0}, \cdots, \frac{f_m}{f_0} \right)(z) \right| \leq \epsilon, \quad \forall z \in U_j.$$

We choose $m \geq n$ and $f_1 = z_1, \cdots, f_n = z_n$. It follows from the above that $F_{i,j}^m$ is a biholomorphic map from each $\tilde{\phi}_i(\vartheta_j(U_j))$ onto its image which contains a ball of radius close to 8 in the cone $\mathbb{C}^{n-1} \times \mathcal{C}'_y$. We will abbreviate $F_{i,j}^n$ by $F_{i,j}$.

For ϵ sufficiently small and i sufficiently large, when restricted to $B_{8s_j r_i}(x_i, \omega_i)$, the map $F^m_{i,j}$ is one-to-one on outside a small tubular neighborhood of $\mathcal{S}_x$. Then by using the above **1** and **2**, one can see that each $F_{i,j}$ is a biholomorphic map from $B_{8s_j r_i}(x_i, \omega_i)$ onto its image which contains the following set

$$U'_j \,=\, \{\,(z', z_n) \,\in\, \mathbb{C}^{n-1} \times \mathcal{C}'_y \,|\, \sqrt{|z'|^2 + |z_n|^{2\bar\beta}} \,<\, 8 - \epsilon\,\}.$$

It follows from the above derivative estimate on $S^b_{i,j}$ that

$$\sup_{B_{8s_j r_i}(x_i,\omega_i)} |dF^m_{i,j}|_{\omega_i} \,\le\, C_m\,(s_j\,r_i)^{-2},$$

where C_m is a constant independent of i and j. This is equivalent to

$$\omega_0 \,\le\, C_m\,(s_j\,r_i)^{-2}\,\omega_i, \tag{7.1}$$

where ω_0 denotes the Euclidean metric on $\mathbb{C}^m$. A consequence of this is that by taking a subsequence if necessary, as i goes to ∞, we get a limiting map

$$F^m_{\infty,j} : B_{8s_j}(y, g_x) \,\mapsto\, \mathbb{C}^m.$$

Moreover, its image is the subvariety $V^m_j \subset \mathbb{C}^m$ which coincides with the limit of $F^m_{i,j}(B_{8s_j r_i}(x_i, \omega_i))$. Such a limit exists because of the well-known Bishop theorem in complex analysis and the following volume bound:

$$\int_{F^m_{i,j}(B_{8s_j r_i}(x_i,\omega_i))} \omega_0^n \,\le\, C_m\,\frac{\mathrm{vol}(B_{8s_j r_i}(x_i,\omega_i))}{(s_j\,r_i)^{2n}} \,\le\, C'_m.$$

The last one follows from the volume comparison.

Next we show that for j sufficiently large, $F_{i,j}(D \cap B_{7s_j r_i}(x_i, \omega_i))$ converge to a local divisor $D^n_j \subset \mathbb{C}^n$. Again it is a corollary of the Bishop theorem, for this purpose, it suffices to bound the volume of $F_{i,j}(D \cap B_{7s_j r_i}(x_i, \omega_i))$. Since $(\mathcal{C}_x, s_j^{-2} g_x, y)$ converge to the standard cone $\mathbb{C}^{n-1} \times \mathcal{C}'_y$ with the standard metric $g_{\bar\beta}$, for j, i sufficiently large, the image of $D \cap B_{8s_j r_i}(x_i, \omega_i)$ under the map $F_{i,j}$ lies in a tubular neighborhood:

$$T_{8,\epsilon} \,=\, \{(z', z_n)\,|\,|z'| < 8, |z_n| < \epsilon\}.$$

On the other hand, using the slicing argument as that in [CCT95], one can show that for each fixed z' with $|z'| < 7.5$, the line segment $\{(z', z_n)\,|\,|z_n| \le 6\}$ intersects with $F_{i,j}(D \cap B_{8s_j r_i}(x_i, \omega_i))$ at k points (counted with multiplicity), where $(1 - \bar\beta) = k(1 - \beta_\infty)$.

It is now easy to bound the volume of $F_{i,j}(D \cap B_{7s_j r_i}(x_i, \omega_i))$: Let $\tilde\eta : \mathbb{R} \mapsto \mathbb{R}$ be a cut-off function such that $\tilde\eta(t) = 1$ for $t \le 7.3$, $\tilde\eta(t) = 0$ for $t > 7.8$ and $|\tilde\eta'| \le 2$, then the volume of $F_{i,j}(D \cap B_{7s_j r_i}(x_i, \omega_i))$ is bounded from above by

$$\begin{aligned} &\int_{F_{i,j}(D\cap B_{8s_j r_i}(x_i,\omega_i))} \tilde\eta(|z'|)\,(\omega_0 + \sqrt{-1}\partial\bar\partial\,|z_n|^{2\bar\beta})^{n-1} \\ \le\ &\int_{F_{i,j}(D\cap B_{8s_j r_i}(x_i,\omega_i))} (\tilde\eta + |z_n|^{2\bar\beta}\tilde\eta')(|z'|)\,\omega_0^{n-1} \,\le\, 3k\,8^{2n}. \end{aligned} \tag{7.2}$$

One can easily see that $F_{\infty,j}(\mathcal{S}_x \cap B_{7s_j}(y, g_x))$ coincides with D_j^n. [23] We can also prove that for any $m > n$, $F_{i,j}^m(D \cap B_{7s_j r_i}(x_i, \omega_i))$ converge to a local divisor $D_j^m \subset V_j^m \subset \mathbb{C}^m$.

For convenience, we summarize the above as follows with one extra property.

Lemma 7.1. *For any $\epsilon > 0$ small, there is a j_ϵ such that for any $j \geq j_\epsilon$, the Lipschtz map $F_{\infty,j}$ maps to $B_{7s_j}(y, g_x))$ into $B_{7+\epsilon}(o, g_{\bar\beta})$ satisfying:*

(1) Its image contains $B_{7-\epsilon}(o, g_{\bar\beta})$;

(2) $F_{\infty,j}(\mathcal{S}_x \cap B_{7s_j}(y, g_x))$ is a local divisor D_j^n which is contained in a tubular neighborhood $T_{8,\epsilon}$;

(3) For any $\delta > 0$, there is an $\epsilon' = \epsilon'(\delta)$ such that $F_{\infty,j}^{-1}(T_{6,\epsilon'}) \subset T_\delta(\mathcal{S}_x) \cap B_{(6+\epsilon)s_j}(y, g_x)$.

Proof. I have shown the validity of (1) and (2). For (3), we can prove by contradiction. If not true, then $F_{\infty,j}^{-1}(V_j^n \cap B_{6.5}(o, g_{\bar\beta}))$ has at least two distinct components, one lies in $\mathcal{S}_x$ while another is not. This implies that for i sufficiently large, the pre-image $F_{i,j}^{-1}(F_{i,j}(D) \cap B_{6.5}(o, g_{\bar\beta})$ has at least two components, which contradicts to the fact that $F_{i,j}$ is one-to-one on $B_{7s_j r_i}(x_i, \omega_i)$. □

Next we observe: For i, j sufficiently large, there are uniformly bounded functions $\varphi_{i,j}$ on $B_{8s_j r_i}(x_i, \omega_i)$ satisfying:

$$(s_j r_i)^{-2}\omega_i = \sqrt{-1}\,\partial\bar\partial\,\varphi_{i,j} \quad \text{on} \quad B_{8s_j r_i}(x_i, \omega_i). \tag{7.3}$$

This is because of the almost constancy of $S_{i,j}^0$. A consequence of this observation is that the volume of $D \cap B_{7s_j r_i}(x_i, \omega_i)$ with respect to $(s_j r_i)^{-2}\omega_i$ is uniformly bounded. In fact, we can prove more.

Lemma 7.2. *We adopt the notations above. Assume that (1) $\xi : \mathbb{R} \mapsto [0, 1]$ is a smooth function with $\xi(t) = 1$ for any $t \geq 8\epsilon$ and (2) f is a holomorphic function on $F_{\infty,j}(B_{7s_j}(y, g_x))$ such that $|f(z', z_n)| \geq |z_n|$ whenever $|z_n| \geq 8\epsilon$. Then there is a uniform constant C such that*

$$s_j^{2-2n} \int_{B_{6s_j}(y,g_x)} |\nabla(h \cdot F_{\infty,j})|^2_{\omega_x}\,\omega_x^n \,\leq\, C \int_{F_{\infty,j}(B_{7s_j}(y,g_x))} \sqrt{-1}\,\partial h \wedge \bar\partial h \wedge \omega_0^{n-1},$$

where $h(z', z_n) = \xi \cdot |f|^2(z', z_n)$ and ω_0 denotes the Euclidean metric on $\mathbb{C}^{n-1}$.

Proof. It suffices to prove the corresponding inequality for each $F_{i,j}$ and then let i go to ∞. As above, let $\tilde\eta : \mathbb{R} \mapsto \mathbb{R}$ be a cut-off function such that $\tilde\eta(t) = 1$ for $t \leq 6.3$, $\tilde\eta(t) = 0$ for $t > 6.8$, $|\tilde\eta'| \leq 2$ and $|\tilde\eta''| \leq 4$, then we have

$$\sqrt{-1}\,\partial\bar\partial\,\tilde\eta(|z'|) \,\leq\, 12\,\omega_0,$$

[23] Here we use the fact that the limit of D coincides with $\mathcal{S}_x$ modulo a subset of Hausdorff codimension at least 4 under the Gromov-Hausdorff convergence of $(M, r_i^{-2}\omega_i, x_i)$ to $(\mathcal{C}_x, \omega_x, o)$. Clearly, the limit lies in $\mathcal{S}_x$. On the other hand, by [CCT95], there is no singular point of $\mathcal{C}_x$ outside the limit of D for which there is a tangent cone of type $\mathbb{C}^{n-1} \times \mathcal{C}'_y$.

moreover, $\tilde{\eta}(|z'|)|dh|^2$ vanishes near the boundary of $F_{i,j}(B_{7s_jr_i}(x_i,\omega_i))$. By the definition of h, we also have

$$\partial h \wedge \partial\bar{\partial} h \,=\, 0.$$

Using these facts and integration by parts, we can deduce

$$\begin{aligned}
& (s_jr_i)^{-2n} \int_{B_{7s_jr_i}(x_i,\omega_i)} \eta(|z'|)\,|\nabla(h\cdot F_{i,j})|^2_{\omega_i}\,\omega_i^n \\
= \;& n \int_{F_{i,j}(B_{7s_jr_i}(x_i,\omega_i))} \eta(|z'|)\,\sqrt{-1}\,\partial h\wedge\bar{\partial}h \,\wedge\, (\sqrt{-1}\,\partial\bar{\partial}(\varphi_{i,j}\cdot F_{i,j}^{-1}))^{n-1} \\
\leq \;& C \int_{F_{i,j}(B_{7s_jr_i}(x_i,\omega_i))} \sqrt{-1}\,\partial h\wedge\bar{\partial}h\wedge\omega_0^{n-1}.
\end{aligned} \tag{7.4}$$

Then the lemma follows.

□

Now we can complete the proof of Lemma 5.8. The arguments are similar to those of the proof for the case with Assumption $\mathbf{A}_1$. For the readers' convenience, we repeat some of them here.

For any small $\epsilon_0 > 0$, since $\bar{\mathcal{S}}_x$ has vanishing Hausdorff measure of dimension strictly bigger than $2n-4$, we can find a finite cover of $\bar{\mathcal{S}}_x \cap B_{\bar{\epsilon}^{-1}}(x,g_x)$ by balls $B_{r_a}(y_a,g_x)$ $(a\,=\,1,\cdots,l)$ satisfying:

(i) $y_a \in \bar{\mathcal{S}}_x$ and $2r_a \leq \epsilon_0$;

(ii) $B_{r_a/2}(y_a,g_x)$ are mutually disjoint;

(iii) $\sum_a r_a^{2n-3} \,\leq\, 1$;

(iv) The number of overlapping balls $B_{2r_a}(y_a,g_x)$ is uniformly bounded.

We denote by $\bar{\eta}$ a cut-off function: $\mathbb{R}\mapsto\mathbb{R}$ satisfying: $0\,\leq\,\bar{\eta}\,\leq\,1$, $|\bar{\eta}'(t)|\,\leq\,2$ and

$$\bar{\eta}(t)\,=\,1 \ \text{ for } \ t\,>\,1.6 \ \text{ and } \ \bar{\eta}(t)\,=\,0 \ \text{ for } \ t\,\leq\,1.1.$$

As before, we set $\chi\,=\,\prod_a \chi_a$, where

$$\chi_a(y)\,=\,\bar{\eta}\left(\frac{d(y,y_a)}{r_a}\right) \ \text{ if } y\in B_{2r_a}(y_a,g_x) \quad\text{and}\quad \chi_a(y)\,=\,1 \ \text{ otherwise.}$$

Then χ vanishes on the closure of $B\,=\,\cup_a B_{r_a}(y_a,g_x)$ which contains $\bar{\mathcal{S}}_x\cap B_{\bar{\epsilon}^{-1}}(x,g_x)$, furthermore, χ satisfies

$$\int_{\mathcal{C}_x} |\nabla\chi|^2\,\omega_x^n \,\leq\, C\,\epsilon_0, \tag{7.5}$$

where C is a uniform constant.

There is a finite cover of $\mathcal{S}_x \cap B_{\bar{\epsilon}^{-1}}(x, g_x)\backslash B$ by balls $B_{6s_b}(y_b, g_x)$ for which Lemma 7.1 holds ($b = 1, \cdots, N$). We may assume that the number of overlapping balls $B_{6s_b}(y_b, g_x)$ is bounded. Choose smooth functions $\{\zeta_b\}$ associated to the cover $\{B_{6s_b}(y_b, g_x)\}$ satisfying:

(1) $0 \leq \zeta_b \leq 1$;

(2) $\mathrm{supp}(\zeta_b)$ is contained in $B_{6s_b}(y_b, g_x)$;

(3) $\sum_b \zeta_b \equiv 1$ near $\mathcal{S}_x \cap B_{\bar{\epsilon}^{-1}}(x, g_x)\backslash B$.

Therefore, $\{\zeta_b\}, 1 - \sum_b \zeta_b$ form a partition of unit for the cover $\{B_{6s_b}(y_b, g_x)\}$ and $B_{\bar{\epsilon}^{-1}}(x, g_x)$.

As before, we denote by η a cut-off function: $\mathbb{R} \mapsto \mathbb{R}$ satisfying: $0 \leq \eta \leq 1$, $|\eta'(t)| \leq 1$ and

$$\eta(t) = 0 \ \text{ for } \ t > \log(-\log \delta^3) \ \text{ and } \ \eta(t) = 1 \ \text{ for } \ t < \log(-\log \delta).$$

For each b, by Lemma 7.1, there is a divisor $D_b^n \subset B_6(0, g_{\bar{\beta}_b})$, where $(1 - \bar{\beta}_b) = k_b(1 - \beta_\infty)$. Choose a local defining function f_b of D_b^n satisfying (2) in Lemma 7.2. We define a function $\gamma_{\bar{\epsilon},b}$ on $B_6(o, g_x)$ as follows: If $|f_b|(y) \geq \bar{\epsilon}/3$, put $\gamma_{\bar{\epsilon},b}(y) = 1$ and if $|f_b|(y) < \bar{\epsilon}$, put

$$\gamma_{\bar{\epsilon},b}(y) = \eta\left(\log\left(-\log\left(\frac{|f_b|(y)}{\bar{\epsilon}}\right)\right)\right). \tag{7.6}$$

Then we put

$$\gamma_{\bar{\epsilon}}(y) = \chi(y)\,(1 - \sum_b \zeta_b(y) + \sum_b \zeta_b(y)\,\gamma_{\bar{\epsilon},b}(y)). \tag{7.7}$$

Clearly, $\gamma_{\bar{\epsilon}}$ is smooth. If we choose ϵ_0 and δ sufficiently small, we have $\gamma_{\bar{\epsilon}}(y) = 1$ for any y with $d(y, \mathcal{S}_x) \geq \bar{\epsilon}$, also $\gamma_{\bar{\epsilon}}$ vanishes in a neighborhood of $\mathcal{S}_x$. Furthermore, by using (7.5), Lemma 7.1 and Lemma 7.2, we can also show

$$\int_{B_{\bar{\epsilon}^{-1}}(o, g_x)} |\nabla \gamma_{\bar{\epsilon}}|^2\,\omega_x^n \leq \bar{\epsilon}.$$

Thus, the proof of Lemma 5.8 is completed.

There are other ways of completing the proof of Lemma 5.8. One is to verify Assumption $\mathbf{A}_1$ (cf. Remark 7.4. Another is to estimate the volume of tubular neighborhood of $\mathcal{S}_x^0$. Let us outline it in the following.

For any small $\epsilon_0 > 0$, we can find a finite cover of $\bar{\mathcal{S}}_x \cap B_{\bar{\epsilon}^{-1}}(x, g_x)$ by balls $B_{r_a}(y_a, g_x)$ ($a = 1, \cdots, l$) with properties (i)-(iv) as above. Then we can have a smooth function χ associated to this covering as we did above. Put $\rho(y) = d(y, \mathcal{S}_x)$ and

$$K = \overline{B_{\bar{\epsilon}^{-1}}(o, g_x)} \backslash \cup_a B_{r_a/2}(y_a, g_x).$$

Define $\gamma_{\bar\epsilon}$ according to (5.10). Clearly, it satisfies (1) and (2) in Lemma 5.8. For (3), if δ is sufficiently small, we only need to prove

$$\int_K |\nabla\eta\cdot\zeta|^2\,\omega_x^n \;=\; \int_K |\eta'\cdot\zeta|^2\,|\nabla\zeta|^2\,\omega_x^n \;\le\; \frac{\bar\epsilon}{2}\,, \tag{7.8}$$

where

$$\zeta(y) \;=\; \log\left(-\log\left(\frac{\rho(y)}{\bar\epsilon}\right)\right).$$

By the well-known co-area formula, we have

$$\int_K |\eta'\cdot\zeta|^2\,|\nabla\zeta|^2\,\omega_x^n \;=\; \int_0^\infty |\eta'(r)|^2\,|\nabla\zeta|\,\mathrm{Vol}(\zeta^{-1}(r)\cap K)\,dr.$$

Clearly, $\zeta(y) \;=\; r$ implies that $\rho(y) \;=\; \bar\epsilon\,e^{-e^r}$, moreover, if we set $s = |\nabla\zeta|^2$, we have

$$\sqrt{s} \;=\; \frac{1}{\bar\epsilon}\,e^{e^r - r}.$$

It is a monotonic function for $r > 0$, thus, the inverse $r = r(s)$ exists. By the co-area formula again, we have

$$\int_0^\infty ds \int_{\{|\nabla\zeta|^2\ge s\}\cap K} |\eta'\cdot\zeta|^2\,\omega_x^n \;=\; \int_0^\infty ds \int_{r(s)}^\infty \frac{|\eta'(r)|^2}{|\nabla\zeta|(r)}\,\mathrm{Vol}(\zeta^{-1}(r)\cap K)\,dr.$$

Exchanging the order of integrals on r and s, we get

$$\int_K |\eta'\cdot\zeta|^2\,|\nabla\zeta|^2\,\omega_x^n \;=\; \int_0^\infty ds \int_{\{|\nabla\zeta|^2\ge s\}\cap K} |\eta'\cdot\zeta|^2\,\omega_x^n.$$

Combining the above integrals, we get

$$\int_K |\nabla(\eta\cdot\zeta)|^2\,\omega_x^n \;=\; \int_0^\infty s'(t)\,dt \int_{\{|\nabla\zeta|^2\ge s(t)\}\cap K} |\eta'\cdot\zeta|^2\,\omega_x^n,$$

where

$$s(t) = \frac{1}{\bar\epsilon^2 t^2\,(-\log t)^2}.$$

Since $\eta'(\zeta(y)) = 0$ unless $\bar\epsilon\,\delta^3 \;\le\; \rho(y) \;\le\; \bar\epsilon\,\delta$, we can deduce from this identity and the following lemma that

$$\int_K |\nabla(\eta\cdot\zeta)|^2\,\omega_x^n \;\le\; C_K\left(\int_0^\delta \bar\epsilon^2 t^2\,s'(t)\,dt \;+\; \int_\delta^\infty \bar\epsilon^2\delta^2\,s'(t)\,dt\right).$$

We get (7.8) from this estimate since the last two integrals tend to 0 as δ goes to 0. Therefore, Lemma 5.8 follows from the following.

Lemma 7.3. *For any compact subset $K \subset \mathcal{C}_x\backslash\bar{\mathcal{S}}_x$, there is a constant C_K such that for any $r < 1$, the volume of $T_r(\mathcal{S}_x)\cap K$ is bounded by $C_K r^2$, where $T_r(\mathcal{S}_x) \;=\; \{z\,|\,d(z,\mathcal{S}_x)\le r\}$.*

This follows from an estimate on the lower bound of the ratio $r^{2-2n}\mathrm{vol}(\mathcal{S}_x \cap B_r(y, g_x))$ for any $r \le 1$ and $y \in K \cap \mathcal{S}_x$. Such an estimate can be easily derived by a blow-up argument and what we have obtained above.

Remark 7.4. *In fact, Assumption* $\mathbf{A}_1$ *can be established by the techniques used above. An approach is to use the maps* $F^m_{\infty,j}$. *One can show that each composition* $F^n_{\infty,j} \cdot (F^m_{\infty,j})^{-1} : V^m_j \mapsto V^n_j$ *is well-defined and the limit of* $F^n_{i,j} \cdot (F^m_{i,j})^{-1}$. *Each such map supposes to be finite and one-to-one on a sufficiently large open subset. One can deduce from these that* $F^n_{\infty,j} \cdot (F^m_{\infty,j})^{-1}$ *is one-to-one. It implies that* $F_{\infty,j}$ *is an one-to-one map. Then we get what we wanted.*

References

[An90] Anderson, M.: Ricci curvature bounds and Einstein metrics on compact manifolds. J. Amer. Math. Soc., **3** (1990), 355-374.

[Au83] Aubin, T.: Réduction du cas positif de l'equation de Monge-Ampère sur les varietes Kahleriennes compactes á la demonstration d'une inegalite. J. Funct. Anal., **57** (1984), 143-153.

[Be11] Berman, R.: A thermodynamical formalism for Monge-Amp'ere equations, Moser-Trudinger inequalities and Kahler-Einstein metrics. Preprint, arXiv:1011.3976.

[Bo11] Bo Berndtsson: Brunn-Minkowski type inequality for Fano manifolds and the Bando-Mabuchi uniqueness theorem. Preprint, arXiv:1103.0923.

[Br11] Brendle, S.: Ricci flat Kähler metrics with edge singularities. Preprint, arXiv:1103.5454.

[Ch03] Cheeger, J.: Integral bounds on curvature elliptic estimates and rectifiability of singular sets. Geom. Funct. Anal. 13 (2003), no. 1, 2072.

[CC95] Cheeger, J. and Colding, T.: Lower bounds on Ricci curvature and the almost rigidity of warped products. Ann. of Math. (2) 144 (1996), no. 1, 189237.

[CCT95] Cheeger, J., Colding, T. and Tian, G.: On the singularities of spaces with bounded Ricci curvature. Geom. Funct. Anal. 12 (2002), no. 5, 873914.

[CN11] Cheeger, J. and Naber, A.: Lower Bounds on Ricci Curvature and Quantitative Behavior of Singular Sets. Preprint, arXiv:1103.1819.

[Co94] Colding, T.: Ricci curvature and volume convergence. Ann. of Math. (2) 145 (1997), no. 3, 477501.

[Do02] Donaldson, S: Scalar curvature and stability of toric varieties. J. Diff. Geom., **62** (2002), 289-349.

[Do10] Donaldson, S: Stability, birational transformations and the Kähler-Einstein problem. Preprint, arXiv:1007.4220.

[Do11] Donaldson, S: Kähler metrics with cone singularities along a divisor. Preprint, arXiv:1102.1196.

[DS12] Donaldson, S and Sun, S: Gromov-Hausdorff limits of Kähler manifolds and algebraic geometry. Preprint, arXiv:1206.2609.

[DT91] Ding, W. and Tian, G.: The generalized Moser-Trudinger inequality, in: Nonlinear Analysis and Microlocal Analysis: Proceedings of the International Conference at Nankai Institute of Mathematics (K.C. Chang et al., Eds.), World Scientific, 1992, 57-70.

[DT92] Ding, W. and Tian, G.: Kähler-Einstein metrics and the generalized Futaki invariants. Invent. Math., **110** (1992), 315-335.

[Fu83] Futaki, A.: An obstruction to the existence of Einstein-Kähler metrics. Inv. Math., **73** (1983), 437-443.

[Gr97] Gromov, M.: Metric structures for Riemannian and non-Riemannian spaces. Edited by J. Lafontaine and P. Pansu.

[JMR11] Jeffres, T., Mazzeo, R. and Rubinstein, Y.: Kähler-Einstein metrics with edge singularities. Preprint, arXiv:1105.5216.

[Li11] Li, Chi: Remarks on logarithmic K-stability. Preprint, arXiv:1104.0428.

[Li12] Li, Chi: Kähler-Einstein metrics and K-stability. Princeton thesis, May, 2012.

[LS12] Li, Chi and Sun, Song: Conic Kähler-Einstein metrics revisited. Preprint, arXiv:1207.5011.

[LX11] Li, Chi and Xu, CY: Special test configurations and K-stability of Fano varieties. Preprint, arXiv:1111.5398.

[Pa08] Paul, S.: Hyperdiscriminant polytopes, Chow polytopes, and Mabuchi energy asymptotics. Ann. of Math. (2) 175 (2012), no. 1, 255296.

[PT06] Paul, S. and Tian, G.: CM stability and the generalized Futaki invariant II. Astrisque No. 328 (2009), 339354.

[Su11] Sun, S.: Note on K-stability of pairs. Preprint, arXiv:1108.4603.

[Ti89] Tian, G.: On Calabi's conjecture for complex surfaces with positive first Chern class. Inv. Math. **101**, (1990), 101-172.

[Ti90] Tian, G.: Kähler-Einstein on algebraic manifolds. Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990), 587598, Math. Soc. Japan, Tokyo, 1991.

[Ti94] Tian, G: Khler-Einstein metrics on algebraic manifolds. Transcendental methods in algebraic geometry (Cetraro, 1994), 143185, Lecture Notes in Math., 1646, Springer, Berlin, 1996.

[Ti97] Tian, G: Kähler-Einstein metrics with positive scalar curvature. Invent. Math., **130** (1997), 1-39.

[Ti98] Tian, G: Canonical Metrics on Kähler Manifolds. Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, 2000.

[Ti09] Tian, G.: Einstein metrics on Fano manifolds. "Metric and Differential Geomtry", Proceeding of the 2008 conference celebrating J. Cheeger's 65th birthday, edited by Dai et al., Progress in Mathematics, volume 239. Birkhäuser, 2012.

[Ti12] Tian, G.: Partial C^0-estimates for Kähler-Einstein metrics. Preprint, August, 2012.

[TW11] Tian, G. and Wang, B.: On the structure of almost Einstein manifolds. Preprint, arXiv:1202.2912.

[TY90] Tian, G. and Yau, S.T.: Complete Kähler manifolds with zero Ricci curvature. I. J. Amer. Math. Soc. 3 (1990), no. 3, 579-609.

[TZ12] Tian, G. and Zhang, Z.L.: Regularity for conic Einstein metrics. In preparation.